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ANNOUNCEMENT

Because there are still a number of calls for the two lithoprinted publications, *Reliability and Validity of Tests* (Ann Arbor, Mich.: Edwards Brothers, 1931) and *Theory of Multiple Factors* (Ann Arbor, Mich.: Edwards Brothers, 1933), which are now out of print, they have been made available on microfilm. Copies may be obtained from the University of Chicago Libraries. They are listed as

Reliability and Validity of Tests
Negative No. 1647 - Approximate cost: 90 cents

Theory of Multiple Factors
Negative No. 1648 - Approximate cost: 35 cents

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THE DISCRIMINANT FUNCTION AND ITS USE IN PSYCHOLOGY

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R. A. Fisher's method of determining a set of weights for the linear combination of the scores in a battery of tests is explained and illustrated. The relationship of this method to combination by multiple regression methods is shown.

The discriminant function of R. A. Fisher should prove to be of value in many problems in experimental as well as in applied psychology. That it has not been more widely used is owing in part, I think, to the lack of an account comprehensible to psychologists who are not especially trained in mathematics. Fisher's papers describing his techniques (2, 3) are not written for the mathematical novice. And psychological writers have usually assumed their readers' acquaintance with the method—a confidence not always well founded.

The present paper is an attempt to explain and to illustrate the discriminant function in such a way that psychologists who are so minded may make use of it. For readers who are unacquainted with the calculus some parts of the mathematical treatment will not be fully understood. But it is hoped that the rationale of the method will, nevertheless, be clear, so that its value will be seen. If this account is understood, the reader may proceed with more confidence to the basic mathematical references.

The purpose of the discriminant function method may be made clear in the following way. Suppose that we have administered a battery of tests to groups of boys and girls, to abnormals and normals, to two vocational groups, or to any two contrasted groups. How best can we *weight* these tests so as to provide a maximum separation or differentiation between our two groups? The equation which yields composite scores under the condition of optimum weighting constitutes a "discriminant function." Such a function is essentially a regression equation from which, once we know the subjects' scores on the tests of the battery, we can estimate their "best" composite scores. When we possess scores in some independent criterion against which our tests may be correlated, the discriminant function becomes an ordinary multiple regression equation as will be shown in Section II.

I. The Discriminant Function when there is no Independent Criterion

The data used in illustrating the calculation of the discriminant function will be found in Table 1. The three tests were taken from a

TABLE 1

Scores Made by 20 Engineer Apprentices (Group A) and 20 Air Pilots (Group B) on Three Tests. Test 1 Is Intelligence; Test 2 the Dynamometer; and Test 3 Perseveration

Group A Tests			Group B Tests		
1	2	3	1	2	3
121	74	254	132	77	249
108	80	300	123	79	315
122	87	223	129	96	319
77	66	209	131	67	310
140	71	261	110	96	268
108	57	245	47	87	217
124	52	242	125	87	324
130	89	242	129	102	300
149	91	277	130	104	270
129	72	268	147	82	322
154	87	244	159	80	317
145	88	228	135	83	306
112	60	279	100	83	242
120	73	233	149	94	240
118	83	233	149	78	271
141	80	241	153	89	291
135	73	249	136	83	311
151	76	268	97	100	225
97	83	243	141	105	243
109	82	267	164	76	264

battery of 6 administered by Travers (10) to two groups. Group A consisted of engineer apprentices, and Group B of air pilots. Test 1 is an intelligence test; Test 2 a dynamometer test; and Test 3 a measure of perseveration. There were 20 men in each group. While the tests and subjects are too few to provide a result of much practical value, the data will serve to illustrate the method and calculations under somewhat simplified conditions. Table 2 shows the mean score on each of the three tests made by Groups A and B and the difference between these means. In terms of the t-test the two groups are significantly different as to mean ability on 2 and 3 but not on 1.

As stated before, our problem is to find a system of weights for Tests 1, 2 and 3 which will provide a maximum separation between the mean composite scores made by the 20 engineers and the 20 air

TABLE 2

Showing the Mean Scores on Tests 1, 2, and 3 Made by Groups A and B and the Significance of the Differences between Means

Test	Group B (means)	Group A (means)	Diff.	t	P(DF = 38)
1	129.3	124.5	4.8	.6566	.50
2	87.4	76.2	11.2	3.3877	<.01
3	280.2	250.3	29.9	3.2215	<.01

pilots. The simplest way in which we can combine our three measures is by means of the following linear (i.e., straight-line) equation:

$$Z = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 , \tag{1}$$

in which Z is the composite or compound score we are seeking; X_1, X_2, and X_3 are the individual scores on Tests 1, 2, and 3 made by the subjects; and λ_1, λ_2, λ_3 (lambdas) are the weights to be assigned these scores. The immediate task is to find values for these λ's which will make the difference between the means of our two groups in the composite measure as large as possible. For readers familiar with the techniques of analysis of variance, this requirement may be stated as follows: to find weights which will maximize the ratio obtained by dividing the variance *between* the composite test means of our two occupational groups by the variance *within* the scores earned by the two groups.

Equation (1) can be written in deviation form for Group A as follows:

$$z = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 , \tag{2}$$

in which the z and the x's represent deviations from their respective means. The variance of Z for Group A may then be found from the following sum of squares and products:

$$\begin{aligned} S = \sum z^2 = \lambda_1^2 \sum x_1^2 + \lambda_2^2 \sum x_2^2 + \lambda_3^2 \sum x_3^2 \\ + 2 \lambda_1 \lambda_2 \sum x_1 x_2 + 2 \lambda_1 \lambda_3 \sum x_1 x_3 + 2 \lambda_2 \lambda_3 \sum x_2 x_3 . \end{aligned} \tag{3}$$

For Group B we may write an equation analogous to (2) in deviation form as follows:

$$z' = \lambda_1 x'_1 + \lambda_2 x'_2 + \lambda_3 x'_3 , \tag{4}$$

in which the primes serve to distinguish deviations around Group B means from deviations around Group A means. The variance within Group B is then

$$\begin{aligned} S' = \sum z'^2 = \lambda_1^2 \sum x'^2_1 + \lambda_2^2 \sum x'^2_2 + \lambda_3^2 \sum x'^2_3 \\ + 2 \lambda_1 \lambda_2 \sum x'_1 x'_2 + 2 \lambda_1 \lambda_3 \sum x'_1 x'_3 + 2 \lambda_2 \lambda_3 \sum x'_2 x'_3 . \end{aligned} \tag{5}$$

Now to obtain the variance *within occupations* for the two groups combined, we must add (3) and (5) and divide by the degrees of freedom obtained by adding the DF contributed by each group. The combined sums of squares and products may be written

$$S = \sum_{p=1}^{3} \sum_{q=1}^{3} \lambda_p \lambda_q S_{pq}, \tag{6}$$

in which S_{pq} represents a sum of squares (e.g., $\sum x_1^2$) or a product (e.g., $\sum x_1 x_3$) depending upon the subscripts. The reader may satisfy himself that (6) is actually the sum of (3) and (5) in the following way. Let p take the value 1 while q takes the values 1, 2, 3 in succession. Then let p take the value 2 while q becomes 1, 2, and 3 in order. Finally, let p take the value 3 while q is successively 1, 2, and 3. The sum of these terms when divided by the appropriate DF gives the *within* variance for the combined groups in the composite measure Z.

The difference between the composite means (i.e., the compound weighted scores) of Groups A and B may be called D. This value D is found from the equation:

$$D = \lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 D_3, \tag{7}$$

in which the d's are the differences between test means given in Table 2, and the λ's are the weights we are seeking. D^2 is proportional to the

TABLE 3

(The Subscripts 1, 2, and 3 Indicate the Three Tests Described in Table 1)

	Σx_1^2	Σx_2^2	Σx_3^2	$\Sigma x_1 x_2$	$\Sigma x_1 x_3$	$\Sigma x_2 x_3$
Group A	7301.00	2301.20	8934.20	1545.00	1575.00	— 77.20
Group B	13068.20	2106.80	23821.21	— 991.40	8344.80	—2125.60
Total	20369.20	4408.00	32755.41	553.60	9919.80	—2202.80

TABLE 4

Sums and Products for Groups A and B

(Subscripts Refer to the Tests 1, 2, and 3)

	1	2	3
1	20369.2 S_{11}	553.6 S_{12}	9919.8 S_{13}
2	553.6 S_{21}	4408.0 S_{22}	—2202.8 S_{23}
3	9919.8 S_{31}	—2202.8 S_{32}	32755.4 S_{33}

variance between the means of our two groups in the composite measure.* Hence, the composite or compound scores which will best discriminate between our two groups are those in which the ratio D^2/S is a maximum. S, it must be remembered, is the sum of squares and products from which the within group variance is calculated.

In order to make the ratio, D^2/S, a maximum we must resort to the calculus. The reader whose calculus is rusty or non-existent will find a good elementary presentation of maxima and minima and of partial differentiation in Chapter I of the text by Peters and Van Voorhis (7), and in the little book by Sylvanus Thompson (9). First, taking logarithms, $\log D^2/S = 2 \log D - \log S$. S is given by equation (6) and D by equation (7). Now if we differentiate the expression $2 \log D - \log S$ with regard to λ (the weights for which we are seeking) and put the result equal to zero, we have

$$\frac{2}{D}\frac{\delta D}{\delta \lambda} - \frac{1}{S}\frac{\delta S}{\delta \lambda} = 0, \quad \text{or} \quad \frac{\delta S}{\delta \lambda} = \frac{2S}{D} \cdot \frac{\delta D}{\delta \lambda}, \tag{8}$$

a function which will be maximal for the values of λ which satisfy this equation. The expressions for S and for D in (6) and (7) are differentiated with respect to λ_1, λ_2, and λ_3 in order. This process, which is called partial differentiation, yields finally the following equations:

$$\begin{aligned} \lambda_1 S_{11} + \lambda_2 S_{12} + \lambda_3 S_{13} &= d_1, \\ \lambda_1 S_{21} + \lambda_2 S_{22} + \lambda_3 S_{23} &= d_2, \\ \lambda_1 S_{31} + \lambda_2 S_{32} + \lambda_3 S_{33} &= d_3, \end{aligned} \tag{9}$$

The expressions in S are not strictly equal to the d's since we have ignored the constant multiplier $2S/D$. But they are *proportional*, so that the d's will bear the same relationship to one another as they would have had had we multiplied by $2S/D$.

Table 3 gives the sums of squares and of products for the three tests calculated separately for Groups A and B; and Table 4 gives these squares and products set out more conveniently. It should be noted that the entries in Table 4 show the sums of squares and of products from which we compute the *within* variance, not the total variance for the two groups combined. The figures in Table 4 represent a combination of sums and products for the two groups when deviations are taken from the means of Group A and of Group B *separately*. Total variance for the two groups would have to be calculated by taking deviations from the general means obtained by throwing the two groups together.

* See Appendix I.

Substituting appropriate values from Table 4, we may write the equations in (9) as follows:

$$\begin{aligned} 20369.2\,\lambda_1 + 553.6\,\lambda_2 + 9919.8\,\lambda_3 &= 4.8\,, \\ 553.6\,\lambda_1 + 4408.0\,\lambda_2 - 2202.8\,\lambda_3 &= 11.2\,, \\ 9919.8\,\lambda_1 - 2202.8\,\lambda_2 + 32755.4\,\lambda_3 &= 29.9\,. \end{aligned} \tag{10}$$

These equations may be solved simultaneously for λ_1, λ_2, and λ_3 by eliminating λ_3 from equations (1) and (2); then from equations (2) and (3), and solving the resulting equations for λ_1 and λ_2.* When this is done, we get the following values of λ:

$$\begin{aligned} \lambda_1 &= -.0004712\,, \\ \lambda_2 &= .0032368\,, \\ \lambda_3 &= .0012731\,. \end{aligned}$$

These weights are very small decimals and for convenience we may multiply each by 1000 and divide by .4712. This done, our λ's become

$$\begin{aligned} \lambda_1 &= -1.0000\,, \\ \lambda_2 &= 6.8692\,, \\ \lambda_3 &= 2.7018\,. \end{aligned}$$

The difference between the means of the two groups with respect to the composite (Z) may now be found by substituting for the λ's in (7) as follows:

$$D = (-1.0000)\,(4.8) + (6.8692)\,(11.2) + (2.7018)\,(29.9) = 152.92\,.$$

To test the significance of this difference of 152.92, we must proceed in the following way. First, the new values of λ are substituted in (10) above to give the totals 10234.905, 23774.309, and 63447.266. These values are, of course, larger than the original d's since the λ-weights have been greatly increased by the multipliers 1000 and 2.12224 (reciprocal of .4712). [If the reader will substitute the original λ's in equation (10), the d's will come out as 4.8, 11.2, and 29.9, the calculated values].

When the three values, 10234.905, 23774.309, and 63447.266 are multiplied, in turn, by λ_1, λ_2 and λ_3, we get S:

$$\begin{array}{r} -\ 10234.905 \\ 163310.483 \\ 171421.823 \\ \hline S = 324497.401 \end{array}$$

* See Appendix II.

which is the sum of squares (and products) for composite scores within the two occupational groups. The reader will note that in order to get this value of S it is necessary to multiply the equations of (10) by λ_1, λ_2, and λ_3 in order, as one of the λ's was removed in (9) by the process of differentiation. Table 5 provides an analysis of total variance for the combined groups into *within* and between variance, the scores being those of the weighted composite. The between sum of squares is equal to $10D^2$ and hence is readily calculated.*

TABLE 5

Source	DF	Sums of Squares	Variance	F	.01 point of F
Between Composite Test Score Means	3	233845.26 ($10\,D^2$)	77948.42	8.65	4.40
Within Composite Test Scores	36	324497.40	9013.82		
Total	39	558342.66			

$$s^2_D = \frac{2(9013.82)}{20}$$

$$t = \frac{152.92}{30.02} = 5.09 \left\{ \begin{array}{l} \mathrm{DF} = 36 \\ P < .01 \end{array} \right. \qquad s_D = 30.02$$

In Table 5 the F-test (between variance divided by within variance)** gives a ratio of 8.65. Since the .01 point for a ratio based upon the given number of DF is 4.40, the result is highly significant and the two groups are clearly separated in terms of the mean scores on the composite. The standard error of the difference between the two means is 30.02, and the t-test verifies the highly significant difference between the group means.

Table 6 presents the same analysis as that shown in Table 5, except that the λ's are expressed in original units. In Table 5, it will

TABLE 6

Source	DF	Sums of Squares	Variance	F	.01 point of F
Between Composite Test Score Means	3	.051926 ($10D^2$)	.017309	8.65	4.40
Within Composite Test Scores	36	.07206 (D)	.002002		
Total	39	.12398 [$D(1 + 10D)$]			

* See Appendix I.

** Three DF are assigned to between means variance, the number of variables which enters into the composite. This leaves 36 DF for within groups variance. See Snedecor (8).

be remembered, the λ's have been multiplied by 1000 and by 2.12224 (the reciprocal of .4712). Note that the *within* sum of squares in Table 6 is equal to the difference between the composite means. The between sum of squares equals D^2; and the total sum of squares equals $D(1 + 10\ D)$. These relations furnish useful checks when the variances are calculated from the data directly.

The equation for estimating composite scores for the men in Groups A and B, found by substituting our values of λ in (1), may be written.

$$Z = -1.000\ X_1 + 6.8692\ X_2 + 2.7018\ X_3\ .$$

Table 7 gives the composite scores for the members of the two groups (each score divided by 10) as estimated from this equation. The difference between the means of the composites is 15.29, which checks the result ($15.29 \times 10 = 152.9$) got from equation (7). As a further

TABLE 7

Composite Scores* from Tests 1, 2, and 3 for Groups A and B Obtained from the Equation $Z = -1.000\ X_1 + 6.8692\ X_2 + 2.7018\ X_3$

	Group A	Group B
	107.36	106.97
	125.21	127.07
	107.82	139.23
	94.11	116.68
	105.29	127.35
	94.55	113.69
	88.70	134.80
	113.52	138.22
	122.45	131.39
	108.97	128.63
	110.29	124.70
	107.55	126.19
	105.40	112.40
	101.10	114.51
	108.17	111.90
	105.97	124.46
	103.92	127.44
	109.51	119.78
	112.97	123.68
	117.57	107.13
Sums:	2150.43	2456.22
Means:	107.52	122.81

$$\left.\begin{array}{r}\text{Diff. between Means} = 15.29\\ s_M = 3.00\end{array}\right\}\ t = 5.10$$

* In the table each composite score has been divided by 10.

check on the significance of the difference between Group A and Group B means, the standard error of the difference was calculated from the data of Table 7. This standard error of 3.00 (30.02 ÷ 10) checks closely the standard error obtained from the within tests variance in Table 5.* Table 8 presents a comparison of the distributions in the composite measure for Groups A and B:—the composite scores have been rounded off to two decimals for convenience. Note that while

TABLE 8

Distribution of Scores Made by Groups A and B in the Composite *Z*

Z	Group A	Group B	Total
139–141		1	1
136–138		1	1
133–135		1	1
130–132		1	1
127–129		4	4
124–126	1	4	5
121–123	1	0	1
118–120	1	1	2
115–117	0	2	2
112–114	2	3	5
109–111	3	0	3
106–108	5	2	7
103–105	3		3
100–102	1		1
97–99	0		0
94–96	2		2
91–93	0		0
88–90	1		1
	20	20	40

the two groups are well separated as to mean score, there is still considerable overlap in the two distributions.

II. The Discriminant Function When an Independent Criterion is Set Up

The same result obtained in Section I above, in which there were no criterion scores, may be found by assuming criterion measures in accordance with methods devised by R. A. Fisher (2). We may, for instance, let each member of Group A be assigned a "score" of 1, and

* It should be noted that to obtain the *SD*'s of the composite measures for Groups A and B, we must divide the sum of squares by $16(n-4)$ instead of by $19(n-1)$. Three DFs are lost because the composite scores are derived from 3 independent variables, for each of which 1 DF must be subtracted.

each member of Group B a "score" of 0 [see Lorge (6) for a good discussion of this procedure]. Then if we let n_1 represent the number of subjects in Group A, and n_2 the number of subjects in Group B (N being equal to $n_1 + n_2$), the mean of the *combined* group is n_1/N. The variance of these scores around this general mean is $n_1 n_2/N^2$ (6). And the correlation with this criterion of any Test X, which has been given to both groups, will be

$$r_{cx} = \frac{(\overline{X}_B - \overline{X}_A)}{N \sigma_x} \cdot \sqrt{n_1 n_2}, \tag{11}$$

in which $\overline{X}_A$ is the mean of the scores in Text X made by Group A, and $\overline{X}_B$ is the mean of the scores in Test X made by Group B. The standard deviation (σ_x) is the standard deviation of the scores in X of A and B treated as one group. The standard deviation of our Test 1 (intelligence) around the general mean (i.e., 126.9) for Groups A and B combined is 22.98. The standard deviation of Test 2 (dynamometer) around its general mean of 81.8 is 12.04; and the standard deviation of Test 3 (perseveration) around its general mean of 265.3 is 32.70. Substituting these values, and the values of n_1, n_2, and N in equation (11), we find the following correlations of our three tests with the criterion to be:

$$r_{cx_1} = .1044; \qquad r_{cx_2} = .4651; \qquad r_{cx_3} = .4587.$$

The intercorrelations of the three tests are found from Table 3 to be

$$r_{12} = .1010; \qquad r_{13} = .3874; \qquad r_{23} = .0741\,.$$

Following the scheme outlined by Garrett (5), we calculate the regression equation* to be

$$\overline{X}_0 = -.00275\, X_1 + .01833\, X_2 + .007421\, X_3 - 2.648\,.$$

If we divide through by .00275, this equation becomes

$$\overline{X}_0 = -1.0000\, X_1 + 6.868\, X_2 + 2.700\, X_3 - 963\,,$$

which checks closely the equation given on page 72 of Section I. This result shows the discriminant function to be essentially a multiple regression equation. In Section I the difference between the means of the two groups was maximized, while in Section II the multiple correlation between the three-test battery and the criterion was maximized.

* For more than 3 variables some systematic solution like the Doolittle scheme should be used. See Peters and Van Voorhis (7).

Further evidence of the identity of the two methods may be found in Table 9. In this table the total variance of the composite (here the regressed) scores has been separated into the variance

TABLE 9

Source	DF	Sums of Squares	Sums of Squares	Variance	F	.01 point of F
Regression	3	$R^2(\frac{n_1 n_2}{N})$	4.186	1.396	8.65	4.40
Remainder	36	$(1-R^2)(\frac{n_1 n_2}{N})$	5.814	.1614		

attributable to regression and the variance attributable to errors of estimate (remainder). A good reference for the method is Snedecor (8). The sum of squares assigned to "regression" equals $R^2(\frac{n_1 n_2}{N})$. R, the multiple correlation coefficient is .644,** while $n_1 n_2/N$, the sum of squares from the criterion mean, is 10. From $R^2(\frac{n_1 n_2}{N})$ we may compute the variance and the standard deviation of the estimated or "regressed" scores. The sum of squares for errors of estimate [i.e., $(1 - R^2)\frac{(n_1 n_2)}{N}$], when divided by the appropriate DF, gives the standard error of estimate—that is, the probable divergence of the regressed scores from the criterion scores. The F-test is made by dividing the variance for regression by the error variance; it gives the same result as that found in Table 5 when between variance was divided by within variance. It is evident, as noted above, that the final composite scores (which are here estimated or regressed scores) separate sharply the means of Groups A and B.

III. Comparative Efficiency of Weighted and Unweighted Test Scores

The critical reader may well ask whether we could not have separated our two groups significantly if scores had been combined di-

** R may be found from the relation $R^2 = \frac{10D}{1 + 10D}$ (See Table 6). By this formula $R = .647$. The R of .644, given above, was computed directly from the data.

rectly, i.e., without weighting, instead of with weights determined by the discriminant function or from the multiple regression equation. This is a fair question, and in order to answer it the three tests taken by Groups A and B were combined in the following way. The 40 scores made by the members of both groups on Test 1 were transmuted into a distribution the mean of which was set at 50 and the standard deviation at 10. And the 40 scores on Tests 2 and 3 were also transmuted into the same distribution. The three scores for each man in Group A were then averaged, and these composites compared with composites determined in the same way for Group B. Strictly speaking, of course, it is inaccurate to say that the test scores were "unweighted." In fact, they are weighted, as the tests are known to be correlated; however, we do not know what the weights are. Results appear in Table 10.

TABLE 10

Group A (Mean)	Group B (Mean)	Diff.	s.e. (diff.)	t	P
46.14	53.86	7.72	2.01	3.84	less than .01

In terms of our composite, it is clear that the two groups are significantly differentiated; and this may seem to dispose of the need for calculating a discriminant function. The result found in Table 10 was really to be expected, however, as inspection of Table 2 shows that Groups A and B were clearly separated by Tests 2 and 3, though not by Test 1. A fairer comparison of the differentiating ability of the discriminant function and the "unweighted" composite is afforded by examining the critical ratios found by the two methods. The critical ratio when the discriminant function is used is 5.10 as compared with a critical ratio of 3.84 for "unweighted" scores. Both *CR*s are significant in the present case; but the latter is only about 75% of the former. If the *CR* obtained from the "unweighted" score composite has been on—or just below—the borderline of significance, use of the discriminant function might have led to a conclusive result. In cases like this the discriminant function can be expected to provide a sharper differentiation than that given by any other system of weighting.

APPENDIX I

To show that D^2 is proportional to the *between* means variance.

Let us suppose that test X has been administered to two groups A and B, each group consisting of 20 individuals. We may compute

the *between* means variance in the following way [Snedecor (8) is a good reference].

1. Correction term, C, $= \dfrac{[\sum X_A + \sum X_B]^2}{40}$.

2. Between means sum of squares =

$$\frac{(\sum X_A)^2 + (\sum X_B)^2}{20} - C. \qquad (12)$$

3. $D = \dfrac{\sum X_A - \sum X_B}{20}$ by definition, and

$$D^2 = \frac{(\sum X_A)^2 - 2\sum X_A \cdot \sum X_B + (\sum X_B)^2}{400}.$$

4. Equation (12) may be written, on simplifying the algebra, as

$$\frac{(\sum X_A)^2 - 2\sum X_A \cdot \sum X_B + (\sum X_B)^2}{40}.$$

Hence, the between means sum of squares equals 10 D^2, and is proportional to D^2.

APPENDIX II

When there are more than three tests, the solution of more than three simultaneous equations for the λ's becomes an extremely tedious job. Fisher (4) has outlined a method for use in the solution of partial regression equations which is applicable in the present case. It, too, is lengthy, but not perhaps so involved as is the solution of many simultaneous equations. Fisher's method requires the use of matrix algebra, a topic unfamiliar to many psychologists. The clearest presentation of matrix algebra for the beginner will be found in Thurstone's Introduction to his *The Vectors of Mind* (11). Bôcher (1) is a straightforward, but more advanced treatment.

In presenting Fisher's method, we begin with the equations of (9) on p. 69

$$\lambda_1 S_{11} + \lambda_2 S_{12} + \lambda_3 S_{13} = d_1,$$
$$\lambda_1 S_{21} + \lambda_2 S_{22} + \lambda_3 S_{23} = d_2,$$
$$\lambda_1 S_{31} + \lambda_2 S_{32} + \lambda_3 S_{33} = d_3.$$

These three simultaneous equations may be arranged in matrix form as follows:

$$\begin{Vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{Vmatrix} \times \begin{Vmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{Vmatrix} = \begin{Vmatrix} d_1 \\ d_2 \\ d_3 \end{Vmatrix} . \tag{13}$$

The first matrix is designated S, the two vertical matrices as λ and d. When S is multiplied by λ we get d [for matrix multiplication, see Thurstone (**11**, pp. 15-21)].

We may now write (13) in the new terminology (matrix notation) as

$$S\lambda = d. \tag{14}$$

Now our object is to solve (14) for the λ's. It is known [see Thurstone (**11**, pp. 23-26)] that if the matrix S is multiplied by its *inverse* (i.e., S^{-1}) the result is an identity matrix the value of which is unity. Multiplying both sides of equation (14) by S^{-1}, we have

$$S\,S^{-1}\,\lambda = S^{-1}\,d$$

or

$$\lambda = S^{-1}\,d \quad (\text{since } S\,S^{-1} = I). \tag{15}$$

Equation (15) tells us that once S^{-1}, the inverse of S, is found, the λ's may be calculated by multiplying S^{-1} by the d's. Our problem, then, is to find S^{-1}. Fisher's method is to put $S\,\lambda = I$ (identity matrix), and solve for λ. Since $SS^{-1} = I$, these values of λ constitute the inverse matrix S^{-1}.

To illustrate, we must solve the three simultaneous equations in (9) three times, letting them equal 1, 0, 0; 0, 1, 0; 0, 0, 1. Thus

$$\begin{aligned} \lambda_1 S_{11} + \lambda_2 S_{12} + \lambda_3 S_{13} &= 1\ 0\ 0\,, \\ \lambda_1 S_{21} + \lambda_2 S_{22} + \lambda_3 S_{23} &= 0\ 1\ 0\,, \\ \lambda_1 S_{31} + \lambda_2 S_{32} + \lambda_3 S_{33} &= 0\ 0\ 1\,. \end{aligned}$$

The solution of these three sets of equations gives three values for λ_1, three values for λ_2, and three values for λ_3. These numbers constitute the inverse matrix S we are seeking. From the equations in (10), we get (to 4 decimals):

$$\begin{array}{cccc} & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & .0588 & -.0169 & -.0189 \\ \lambda_2 & -.0169 & .2358 & .0216 \\ \lambda_3 & -.0189 & .0216 & .0384\,. \end{array}$$

When the columns of this matrix are multiplied by d_1(4.8), d_2(11.2), and d_3(29.9) in order, we get, on summing

$$\begin{aligned} \lambda_1 &= -\ .000472\,, \\ \lambda_2 &= \ .003240\,, \\ \lambda_3 &= \ .001290\,, \end{aligned}$$

and these figures check closely the values for λ given on page 70.

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CONTRIBUTION TO THE MATHEMATICAL THEORY OF HUMAN RELATIONS: VI. PERIODIC FLUCTUATIONS IN THE BEHAVIOR OF SOCIAL GROUPS

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A case of interaction between two groups of "active" and one group of "passive" individuals, in which the efforts of the influencing active groups decrease with increasing total success in the past, is studied. In that case the numbers of passive individuals, exhibiting respectively the two opposite behaviors, fluctuate periodically, with a positive damping.

In previous papers (1, 2, 3), we have studied the theory of various cases of interaction of two groups of "active" individuals, exhibiting mutually excluding behaviors, with groups of "passive" ones. In this paper we shall study another such case, which presents some interest.

We have previously (1) considered the case in which the effort of the active individuals decreases as the instantaneous success of this effort increases. In other words, the more passive individuals are converted to behavior A, the less will be the efforts of the active individuals of type A.

Let us consider here the case in which the increase of the effort is determined not by the instantaneous value of the attained success, but by the integral success over all of the past time. In other words, the total success in the past influences the effort of the active individual.

The more remote the success in the past, the less we would expect it to affect the effort of the individual. Using the same notations as before (1), and denoting by $f_1(t-\tau)$ and $f_2(t-\tau)$ two functions of the present time t and a past time τ, of such a nature that $f_1(t-\tau)$ and $f_2(t-\tau)$ decrease with increasing $t-\tau$, we may express now the coefficients of influence a_0 and c_0 in the form:

$$a_0(t) = a_0^*[1 - \varepsilon \int_{-\infty}^{t} x(\tau) f_1(t-\tau)\, d\tau]\,;$$
$$c_0(t) = c_0^*[1 - \varepsilon' \int_{-\infty}^{t} y(\tau) f_2(t-\tau)\, d\tau]. \tag{1}$$

Instead of equation (2) of (1), we now have

$$\frac{dx}{dt} = a_0^*[1 - \varepsilon \int_{-\infty}^{t} x(\tau) f_1(t-\tau)\, d\tau]\, x_0 + ax$$
$$- c_0^* [1 - \varepsilon' \int_{-\infty}^{t} y(\tau) f_2(t-\tau)\, d\tau]\, y_0 - ay\,, \tag{2}$$

or, because of

$$x + y = N'\,, \tag{3}$$

$$\frac{dx}{dt} = a_0^*[1 - \varepsilon \int_{-\infty}^{t} x(\tau) f_1(t-\tau)\, d\tau]\, x_0 + 2ax - c_0^*\{1 - \varepsilon' \int_{-\infty}^{t} [N'$$
$$- x(\tau)] f_2(t-\tau)\, d\tau\}\, y_0 - aN'. \tag{4}$$

As an illustration, we shall solve equation (4) for a particularly simple case, namely

$$f_1(t-\tau) = f_2(t-\tau) = e^{-\alpha(t-\tau)}, \tag{5}$$

where α is a constant.

Equation (4) may be written:

$$\frac{dx}{dt} = a_0^* x_0 - a_0^* x_0\, \varepsilon\, e^{-\alpha t} \int_{-\infty}^{t} x(\tau) e^{\alpha\tau}\, d\tau - c_0^*\, y_0$$
$$+ c_0^*\, y_0\, \varepsilon'\, e^{-\alpha t} \int_{-\infty}^{t} [N' - x(\tau)] e^{\alpha\tau}\, d\tau + 2ax - aN'. \tag{6}$$

Multiplying first by $e^{\alpha t}$, then differentiating with respect to t, remembering that

$$\frac{d}{dt} \int_{-\infty}^{t} x(\tau) e^{\alpha\tau}\, d\tau = x(t) e^{\alpha t},$$

then shortening everything by $e^{\alpha t}$ and rearranging, we find

$$\frac{d^2x}{dt^2} + (\alpha - 2a)\frac{dx}{dt} + (a_0^* x_0 \varepsilon + c_0^* y_0 \varepsilon' - 2\alpha a)\, x$$
$$- (\alpha a_0^* x_0 - \alpha c_0^* y_0 + c_0^* y_0 \varepsilon' N' - \alpha a N') = 0\,. \tag{7}$$

Let

$$a_0^* x_0 \varepsilon + c_0^* y_0 \varepsilon' - 2\alpha a = A\,,$$
$$\alpha a_0^* x_0 - \alpha c_0^* y_0 + c_0^* y_0 \varepsilon' N' - \alpha a N' = B\,. \tag{8}$$

If

$$0 < (\alpha - 2a)^2 < 4A\,, \tag{9}$$

then equation (7) represents a damped oscillation around a value $x = \bar{x}$, of the form:

$$x = \bar{x} + c_1\, e^{-(\alpha/2-a)t} \sin(\nu t + \delta), \tag{10}$$

where

$$\nu = \tfrac{1}{2}\sqrt{(\alpha - 2a)^2 - 4A}\,, \tag{11}$$

while c_1 and δ are determined by the initial conditions. The value $\bar{x}$ of x around which it oscillates and to which it tends as the amplitude decreases, is

$$\bar{x} = \frac{B}{A} = \frac{\alpha a_0{}^* x_0 - \alpha c_0{}^* y_0 + c_0{}^* y_0 \varepsilon' N' - \alpha a N'}{a_0{}^* x_0 \varepsilon_0 + c_0{}^* y_0 \varepsilon' - 2\alpha a}. \tag{12}$$

If (10) is not satisfied, the value (12) is approached aperiodically.

Thus for constant values of x_0 and y_0, the values of x and y, and hence of x/y, will fluctuate periodically. While for some values of the constants x may always remain less than y, for some other values x may periodically exceed y and again drop below it. Thus the type of behavior characteristic of the "majority" of the passive population will periodically fluctuate.

When α tends to infinity, then $e^{-\alpha(t-\tau)}$ is large only for values of τ in the neighborhood of t. But the value of $\int_{-\infty}^{t} x(\tau)\, e^{-\alpha(t-\tau)} d\tau$ tends to zero. The case reduces to that of equation (6) of (1). And, indeed, making $\alpha = \infty$, we obtain from (12):

$$x = \frac{aN' + c_0{}^* y_0 - a_0{}^* x_0}{2a}. \tag{13}$$

The numerator of (13) is identical with the expression in parentheses in equation (6) of (1). When this expression is positive, equation (6) of (1) has an equilibrium configuration given by (13). If it is negative, then (13) means physically that $x = 0$, $y = N'$, and we have again the case discussed on page 224 of (1).

When $\alpha = 2a$, the oscillations are undamped. For $\alpha < 2a$ we have a negative damping, the amplitude increasing indefinitely. Since both x and y are finite and cannot exceed N', the physical interpretation of this case would be that, for instance, x increases in an oscillatory way until it becomes equal to N'. If x reaches N' just at the moment of the maximum, when $dx/dt = 0$, then it will go down to zero, and reach zero when $dx/dt < 0$. In other words, y will reach N' when $dy/dt > 0$. Due to the symmetry of x and y, we may therefore confine ourselves to the consideration of the case where x attains

the value N' when $dx/dt > 0$. Let that happen at $t = t_1$. From then on x must remain constant, at least for a while, since it cannot increase and it cannot begin to decrease immediately because at that time the right side of equation (6) is positive. As x remains constant, and equal to N', the second term of the right side of (6) will increase, tending for large values of t to $N'a_0^*x_0\varepsilon/\alpha$, while the fourth term tends to zero.

If α is sufficiently small, then after a sufficient time has passed after t_1, so that the second term is near enough to $N'a_0^*x_0\varepsilon/\alpha$ and the fourth to zero, we shall necessarily have

$$a_0^*x_0 - c_0^*y_0 + aN' - \frac{N'a_0^*x_0\varepsilon}{\alpha} < 0. \tag{14}$$

Hence the right side of (6) will become negative and x will begin to decrease. If

$$\bar{x} \backsim \frac{N'}{2} \tag{15}$$

then x will swing now into the other extreme of $x = 0$, $y = N'$. The same argument applies to that situation, except that the role of the second and fourth terms are interchanged. If α is so small that

$$a_0^*x_0 - c_0^*y_0 - aN' + \frac{c_0^*y_0\varepsilon'N'}{\alpha} > 0, \tag{16}$$

then after remaining for a while equal to zero, x will increase again to the value of N', provided $\bar{x}$ is in the neighborhood of $N'/2$. Thus if α is so small that both (14) and (16) hold, and if inequality (16) holds, the behavior of the passive individuals will be swinging between behavior A and B, back and forth. Except during the transition periods, all individuals for a while will exhibit behavior A, then all for a while will swing to behavior B, and so forth.

If (14) and (15) are satisfied, but (16) is not, then after reaching the value $x = 0$, this value will persist. All individuals will exhibit behavior B. On the contrary, if (15) is satisfied, but (14) is not, x will remain equal to N', once it reaches it. All passive individuals will exhibit behavior A.

If $\bar{x}$ is much larger than $N'/2$, then x, decreasing from $x = N'$ may not reach $x = 0$ before it begins to increase again. The swinging will go on between $x = N'$ and some finite value $x = x_1$: *mutatis mutandis* this holds for the reverse case, when $\bar{x}$ is much smaller than $N'/2$, and when $x = 0$ is reached first.

Oscillations of group behavior from one extreme to another are

actually sometimes observed. It must be emphasized, however, that the assumption (5) is rather artificial and that until more complex cases have been investigated, no practical applications can be made.

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CONTRIBUTIONS TO THE THEORY OF HUMAN RELATIONS: VII. OUTLINE OF A MATHEMATICAL THEORY OF THE SIZES OF CITIES

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In generalization of a previous study, a mathematical approach to the theory of the average size of cities as well as of the distribution of city sizes is outlined.

In previous papers (1, 2) we have developed a theory of increase of the ratio of urban to rural populations and found it in fair agreement with some observations. The purpose of this paper is to go a step further and outline a mathematical approach to the problem of the size of the cities.

In the first approximation we may neglect the distribution of city sizes and speak of the "average" size of cities as if all cities were of that average size. Next we shall treat the problem of the actual distribution of city sizes.

In treating the simple problem of the ratio of the urban to rural population, we considered that the amounts of goods produced by an individual in the city, p_u, and in the country, p_r, depend in general on both N_u and N_r, the total urban and rural populations. We must now somewhat generalize this picture and consider that both p_u and p_r depend not only on N_u and N_r, but also on the average size of the cities or, what amounts to the same thing, on the number m of the cities. This is closer to reality than the original assumption, for the conditions of work in a community will depend in general on the size of that community.

We thus put

$$p_u = f_u(N_u, N_r, m); \quad p_r = f_r(N_u, N_r, m). \tag{1}$$

By the same argument as before (1, 2), we have for equilibrium

$$p_u = p_r. \tag{2}$$

Equation (2), together with

$$N_u + N_r = N, \tag{3}$$

where N is the total population, determines N_u and N_r as functions of N and m; thus,

$$N_u = F_u(N,m), \quad N_r = F_r(N,m). \tag{4}$$

Substituting (4) into (1), we find

$$p_u = p_u(N,m); \quad p_r = p_r(N,m). \tag{5}$$

If each individual tends to choose or modify his surroundings in such a way as to make his p_u (or p_r) as large as possible, then all individuals will aggregate in cities of such a size which makes p_u and p_r a maximum. Hence we have for the determination of m:

$$\frac{dp_u}{dm} = 0 \quad \text{or} \quad \frac{dp_r}{dm} = 0. \tag{6}$$

Because of (2), either of the relations (6) leads to the same expression

$$m = f(N). \tag{7}$$

Equations (4) and (7) determine N_u, N_r, and m as a function of the total population N.

Thus the determination of m in each individual case reduces to the determination of the function f_1 and f_2 in (1). By making different plausible assumptions about these functions, we shall obtain different expressions for m. If $m = f(N)$ is known from observations, we may attempt to determine backwards the form of f_1 and f_2. An important factor in determining f_1 and f_2 is the development of means of transportation. For the larger m, and the smaller therefore the size of the cities, the lesser distance must be covered in transporting rural supplies to the city, which influence the value of p_u. Similarly the ease of supply of the rural population with urban products affects the value of p_r.

We may for instance put

$$\begin{aligned} p_u &= a_1 - a_2 N_u + (a_3 + a_4 m) N_r; \\ p_r &= b_1 - b_2 N_r + (b_3 + b_4 m) N_u, \end{aligned} \tag{8}$$

where a_i and b_i are parameters. a_4 and b_4 may, for instance, be taken as proportional to the total length of railroads and highways per square mile. We find easily

$$\frac{N_u}{N} = A - \frac{B}{N}, \tag{9}$$

where A and B are constants, functions of m. The complete solution of the problem, using expression (8), is elementary but rather cumbersome.

Let us now consider the distribution of city sizes. Let N_i be the total number of people inhabiting all cities of size (population) n_i. In general we shall have for the amount of goods produced per person in such cities:

$$p_i = f(n_i, N_i). \tag{10}$$

The equilibrium we shall have

$$p_1 = p_2 = \cdots = \text{Const.} = p. \tag{11}$$

Hence

$$f(n_i, N_i) = p. \tag{12}$$

Equation (12) gives us N_i as a function of n_i and p.

$$N_i = \bar{f}(n_i, p). \tag{13}$$

p is determined by the requirement

$$\Sigma N_i = N, \tag{14}$$

or

$$\Sigma \bar{f}(n_i, p) = N. \tag{15}$$

For very large numbers we may pass to integrals instead of sums and determine p from

$$\int_0^\infty \bar{f}(n, p)\, dn = N, \tag{16}$$

while equation (13) now reads

$$N(n) = \bar{f}(n, p). \tag{17}$$

As an illustration we may consider, for instance, such an expression as

$$p_i = a_i - b_i N_i, \tag{18}$$

where

$$a_i = f_1(n_1) \quad ; \quad b_i = f_2(n_i). \tag{19}$$

We then find

$$N(n) = \frac{f_1(n) - p}{f_2(n)}, \tag{20}$$

p being determined by

$$\int_0^\infty \frac{f_1(n) - p}{f_2(n)}\, dn = N. \tag{21}$$

As another illustration we may consider

$$p_i = a_i - b_i N_i + (c_i + \sum \frac{N_i}{n_i})(N - N_i)\,;$$
$$a_i = f_1(n_1)\ ;\ b_i = f_2(n_i)\ ;\ c_i = f_3(n_i). \tag{22}$$

Since $\sum(N_i/n_i)$ is the total number of cities, this expression corresponds to expression (8), mentioned above. For the continuous case, we have:

$$p = f_1(n) - f_2(n)N(n) + Nf_3(n) + N\int_0^\infty \frac{N(n)}{n}dn - N(n)f_3(n)$$
$$- N(n)\int_0^\infty \frac{N(n)}{n}dn. \tag{23}$$

Putting

$$K = \int_0^\infty \frac{N(n)}{n}dn, \tag{24}$$

K being the total number of cities, we find from (23)

$$N(n) = \frac{f_1(n) + Nf_3(n) + NK - p}{f_2(n) + f_3(n) + K}. \tag{25}$$

Introducing (25) into (24), we find a relation between p and K; thus

$$K = K(p), \tag{26}$$

so that (25) now becomes

$$N(n) = \frac{f_1(n) + Nf_3(n) + NK(p) - p}{f_2(n) + f_3(n) + K(p)}. \tag{27}$$

The value of p is again determined from

$$\int_0^\infty N(n)dn - N. \tag{28}$$

Applications of those theoretical considerations will be discussed elsewhere.

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SOME FACTORS OF TEMPERAMENT: A RE-EXAMINATION

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By application of the technique presented earlier,* the factors found by the Guilfords in two studies of personality factors are reanalyzed. The rotation technique for the isolation of meaningful factors relates the factors of the two studies to each other through three common items, and to other factors and traits appearing in the literature of personality measurement. For each of four of the factors resulting from the analysis of the first battery there is close agreement with its counterpart from the analysis of the second battery with respect to loadings of the three common items. These factors are interpreted as perseverance, surgency, flexibility, and tension. Other factors appearing in only one or the other of the batteries are tentatively identified.

Introduction

This paper is a sequel to one by the present authors entitled "On the Interpretation of Common Factors: a Criticism and a Statement";* and it attempts in a concrete fashion to illustrate and apply the principles set forth there.

We propose to submit to a fresh analysis some of the contributions made to the study of temperament by J. P. and R. B. Guilford, particularly those contained in two articles which appeared in 1939.** The range of qualities dealt with in these articles is unusually large, the number of subjects is adequate, and there can be little doubt that some of the factors extracted are of value. At the same time it is difficult to accept all the conclusions reached by the authors, and it is probable that the factor pattern can be improved.

The Guilfords follow the practice, criticised in our previous article, of fixing the position of the reference axes on the basis of the conception of simple structure. They give no justification for this procedure, and there is no reason to believe that in the construction of the batteries of tests, they used objective factors as a guide. We have reanalyzed their material, and we believe that by so doing we have

* *Psychometrika*, 1943, 8, 53-64.

** (a) J. P. Guilford and R. B. Guilford. Personality factors, *D*, *R*, *T*, and *A*. *J. abnor. soc. Psychol.*, 1939, 34, 21-36.

(b) J. P. Guilford and R. B. Guilford. Personality factors *N* and *GD*. *J. abnor. soc. Psychol.*, 1939, 34, 239-248.

obtained more significant and more probable results than they reached by their more mechanical procedure. One difficulty, however, may be mentioned. In neither of the two cases have we been provided with the centroid matrix. In their first article the factorial matrix is given on page 26, and, accepting it as correctly derived from the centroid matrix, we have used it as the basis of our analysis. In the second article the factorial matrix is given on page 243, but in the derivation of it arithmetic errors seem to have intruded themselves. The authors extracted seven factors and state that "at this stage the largest residual was .130, which is about the size of the smallest significant coefficient of correlation . . ." This cannot be checked directly, for as we have said, the centroid matrix is not given, but a table of residuals can be constructed from the factorial and correlational matrices. When this is done there are ten residuals over .25, and of these four exceed .40. Obviously there is a mistake somewhere, and as it cannot be located from the data a fresh centroid analysis is necessary. When we undertook this analysis we found that six factors seemed to be sufficient to bring the residuals down to the level of random error, the highest residual, taken irrespective of sign, being .124, and the median .032. The centroid matrix is given in Table 1. It will be understood, there-

TABLE 1

	I	II	III	IV	V	VI	h^2
7	.281	.195	.181	.191	–.134	–.084	.212
10	.463	.247	.190	.070	–.228	–.189	.404
11	.363	–.119	.581	–.367	.154	–.140	.662
20	.327	.373	–.075	.266	.202	.079	.369
24	–.482	.375	.223	–.149	–.058	.140	.467
31	.327	.080	–.132	.052	–.097	–.140	.162
34	.267	.416	–.199	–.100	.093	.117	.316
40	–.315	.232	.216	.150	.127	–.176	.269
41	.353	.277	–.062	–.081	–.050	–.117	.228
43	.061	.164	–.328	–.105	.231	–.033	.204
44	.376	–.169	–.070	.196	–.224	–.019	.264
50	.388	–.288	–.170	.203	.178	.315	.435
60	.375	–.164	.578	–.316	.122	–.160	.642
63	.400	.136	–.150	–.046	.054	–.147	.228
67	.073	–.305	–.208	–.458	–.311	.126	.464
69	.327	.359	.084	–.105	.242	.147	.334
72	.392	–.094	.242	.386	–.088	–.022	.378
74	.474	–.509	–.231	.369	.140	–.079	.699
79	.280	.205	.162	.063	–.114	.220	.212
83	.320	–.441	–.140	.154	.258	–.171	.436
85	–.357	–.298	–.141	–.094	–.166	–.393	.427
89	.320	–.276	–.139	–.203	–.212	.160	.310
95	.177	.218	.042	.255	.076	–.066	.156

fore, that in the case of the Guilfords' first article our rotation starts from their factorial matrix, and in the case of the second from our fresh centroid matrix.

The Tests Common to the Two Batteries

We may begin our discussion with a point raised in our previous article. We said that "if two or more batteries contain a number of tests in common, a hypothesis which gives these tests concordant structural analyses in the different batteries is to be preferred, particularly if, within the limits of random error, it gives them the same factor loadings." The two articles under review deal with separate batteries, but three tests are common to them, viz., 40, 67, and 85. Of the nine common factors found by the first analysis, four enter into at least two of these three variables, and produce the correlations between them. These are I, Depression, II, Rhathymia, III, Liking for Thinking, and IX, an unidentified but not unimportant remainder. In the second article, however, only two factors are found to be common to the same three variables, viz., I, Nervousness, and II, General Drive. Table 2, abstracted from the factorial matrices, shows the significant factor loads of the three tests, only loads of .2 and over being given.

TABLE 2

A—*First Analysis*	*Variables* 40	67	85
Factors			
I Depression	—	–.30	–.36
II Rhathymia	–.55	.76	—
III Liking for Thinking	—	.20	.28
IX Unidentified	–.22	—	–.32
B—*Second Analysis*			
I Nervousness	—	–.33	.55
II General Drive	–.36	.40	—

In the table of the second article, from which these figures are drawn, the sign of the first factor load of variable 85 is given as positive and we print it so, but it is highly probable that it should be negative. But, if so, there is a resemblance between the first factors of the two analyses, although there is no evidence that the authors recognize it. Moreover, a comparison of the loads of the second factors in each case would again give rise to the suspicion of some community between them, and a scrutiny of them in the light of the other variables in the batteries makes it clear that although the two factors are not identical they are definitely related and are not independent of one another. The result then of the total analysis is that the three

vectors are projected on to six axes. Four of these, forming the first set, are orthogonal, and the two forming the second set are also orthogonal. But the relations between the two sets are not defined, although it is certain that some of the angles are oblique.

As the analyses of these three variables stand, we are presented with six common factors, and the problem at once arises can their number be reduced? Is it possible to locate the two axes of the second set within the framework defined by the first set? If so, not more than four common factors will be required here. Such a solution, if feasible, would be in accordance with the principle of parsimony, but we are prevented from finding it if we adhere to the conception of simple structure. Each analysis has its own factors, and the attempt to bring one set into line with another is precluded.

A Positive Method — The Recognition of Factors

We have argued that the most fruitful method in dealing with limited analyses like those under discussion often is to begin by identifying factors already recognized and given some claim to objectivity by previous work. We submit it should be followed here.

A — First Battery

Beginning with the former of the two articles which we are examining, we may consider what common factors the first battery may be expected to contain.

1. The first two factors which the Guilfords isolate arouse misgivings. The first is called Depression, the second "a freedom from care . . . and a lack of serious-mindedness" which they call Rhathymia. It is hardly credible, however, that a tendency to depression and a tendency to be cheerful are independent and unconnected, and it seems probable that these two factors are really projections from a single bi-polar factor. There is a good deal of independent evidence for a factor of this kind. Cattell has called it Surgency,* and we have verified its existence both in Webb's material** and in some of our own. † The easiest way to draw an axis representing this factor is to use as guides items 1 (Are you ordinarily a care-free individual?) and 67 (Would you rate yourself as a happy-go-lucky individual?). We suggest therefore that the first axis should provisionally be passed through the centre of gravity of these two variables.

2. A second factor which almost certainly is present in material

* R. B. Cattell. A Guide to Mental Testing. Chapter 5; and elsewhere.

** H. A. Reyburn and J. G. Taylor. Some factors of personality. A further analysis of some of Webb's data. *Brit. J. Psychol*, 1939, 30, 151-165.

† H. A. Reyburn and J. G. Taylor. Factors in introversion and extraversion. *Brit. J. Psychol.*, 1941, 31, 335-340.

of this kind is Perseverance, the factor named "*w*" by Webb. It is unlikely that any single variable is very heavily loaded with it, but there are several into which it may be expected to enter. These are:—

12. Are you inclined to analyze the motives of others?
15. Are you inclined to worry over possible misfortunes?
25. Would you rate yourself as an impulsive individual?
29. Are you inclined to be over-conscientious?*
30. Do you often crave excitement?
40. Are you inclined to stop and think things over before acting?
68. Do you enjoy thinking out complicated problems?

An axis passed through their centre of gravity, after the appropriate reflections have been made, should not be far from the true position. The variables differ considerably in character, and the other factors in them may be expected to cancel out.

3. In another enquiry** we found a factor which we called Sociability, and it seems not unlike the Guilfords' factor S (Shyness) measured from the opposite end. The axis for this factor can be derived from variables 2 (Do you usually have difficulty in starting a conversation with strangers?), 18 (Are you inclined to keep in the background on social occasions?), and 26 (Do you enjoy getting acquainted with most people?), 2 and 18 being reflected.

4. A scrutiny of the questionnaire itself makes it evident that a common factor, which may be called Alertness, is bound to appear in the material, for there are three questions all directed to the same topic. These are 14 (Do you usually keep in close touch with things going on around you?), 45 (Are you less attentive than most individuals to things going on around you?) and 87 (Are you more alert to your immediate surroundings than the average person?). The centroid of these three variables may therefore be used to locate the fourth axis.

5. Another factor which suggests itself as probable is given by 12 (Are you inclined to analyze the motives of others?), 28 (Do you frequently find yourself in a meditative state?) and 70 (Are you inclined to be introspective?). It is not quite clear, however, what the nature of this factor will turn out to be. It may be something in line with the traditional conception of introversion, in which case it may be called autistic thinking. On the other hand it may be more general, with less reference to the self, and better named Deliberative Thinking. We shall return to this point later.

* No. 29, it may be remarked, turns out not to justify the expectation. Conscientiousness and over-conscientiousness prove to be rather different qualities.

** H. A. Reyburn and J. G. Taylor. Factors in introversion and extraversion. *Brit. J. Psychol.*, 1941, **31**, 335-340.

6. At this stage it becomes evident that the two items dealing directly with thinking, viz., 62 (Do you like to try your wits in solving puzzles?) and 68 (Do you enjoy thinking out complicated problems?), are by no means exhausted. Another factor connected with the intellectual side of life is obviously still left in the matrix, and it may be extracted by utilizing the two items mentioned above.

7. Although the Guilfords did not attempt to identify more than six factors, the number we have already found, it seems possible, in view of the fact that one of our six, viz., Perseverance, has no counterpart in their analysis, to go further. The overlap between 19 (Are you more interested in athletics than in intellectual things?) and 84 (Do you usually become so absorbed in watching an athletic contest that you completely forget yourself?) may be used as a guide to the seventh factor, and it gives a consistent picture.

8. When the foregoing factors have been extracted two residual factors are left. One of the factors which the Guilfords failed to identify, their IX, has significant loads in items 1, 10, 29, 40, and 85. Our eighth axis may be given a somewhat similar position, having significant loads in four of these, viz., 10 (Are you relatively unconcerned with what others think of your actions?), 29 (Are you inclined to be over-conscientious?), 40 (Are you inclined to stop and think things over before acting?) and 85 (Can you relax yourself easily when sitting or lying down?). Two other items have significant loads, viz., 15 (Are you inclined to worry over possible misfortunes?) and 25 (Would you rate yourself as an impulsive individual?). Borrowing a term from June Downey, we might call this Lack of Freedom from Load, or perhaps Tension.

B — Second Battery

We now turn to the second article. The questionnaire employed in the research which it records was designed to test the assumption that important temperamental differences depend on the hyper-activity or hypo-activity of the nervous system.

1. In view of the presence of item 67 in both batteries and the considerable load of Surgency which it carries in the first one, we may provisionally pass an axis through it. This will give us a first approximation to Surgency.

2. In order to locate the factor of Persistence, which is probably also present in the second battery, we may employ items 11 (Do you like to change from one type of work to another frequently?), 20 (Are you easily disturbed by distracting stimuli while doing mental work?), 21 (Do you express such emotions as delight, sorrow, anger and the like readily?), 40 (Are you inclined to stop and think things over first

before acting?) and 60 (Would you like a position in which you changed from one kind of task to another frequently during the day?)

3. A third factor may be extracted, corresponding broadly to that which the Guilfords call Nervousness, and for this purpose we may employ variables 34 (Are you easily startled by unexpected stimuli?), 41 (Do you have any "nervous habits," like chewing your pencil or biting your finger nails?), 43 (Does it annoy you to see a person clean his finger nails in public?) and 63 (While listening to a lecture, does your hand keep active in writing or drawing when not taking notes?).

4. In the first battery we obtained a factor concerned with thinking, but we left its precise nature undetermined. If it is interpreted as deliberative thinking it may appear also in the second battery, in spite of the small amount of direct reference to thinking which the latter contains. The variables involved in deliberativeness of action and behavior, or the reverse, may be used as a guide to it, viz., 24 (Are you inclined to be slow and deliberate in movement?), 50 (Are you inclined to rush from one activity to another without pausing for rest?), 74 (Are you inclined to be quick in your actions?) and 83 (Can you turn out a large amount of work in a short time?).

5. Two factors are left. One of these may be identified provisionally with the eighth factor of the first analysis, which we have called Tension.

6. The sixth factor is not represented in the first analysis and is necessarily determined when the others are located. It is a little difficult to identify it, as it is not present in high degree in many of the variables, but it would seem to be connected with a man's control over himself, and would characterize the inhibited and formal type of person.

Revision of Hypotheses; Further Rotation

The provisional hypotheses on which we have been working may now be tested by considering the results which they give for the vari-

TABLE 3

A	I	II	III	IV	V	VI	VII	VIII	IX
40	–.427	.460	.033	.110	.261	.195	.135	–.343	–.068
67	.855	–.058	–.034	.057	.035	–.006	.034	.020	.143
85	.289	.294	.196	–.060	.008	.142	.248	–.319	–.149

B	I	II	III	IV	V	VI
40	–.395	.132	–.053	–.249	–.158	–.076
67	.681	.000	.000	.000	.000	.000
85	.205	.316	–.012	.033	.454	–.279

ables, 40, 67, and 85, which are common to the two enquiries; and it becomes evident that some revision is desirable. Table 3 shows the loads of the factors in the two cases.

The first factor load on variable 67 is too great in both analyses, for it is highly probable that a thinking factor of some kind is connected with the tendency to be happy-go-lucky or the reverse. Hence it seems desirable to rotate the first axis with another one. In the first analysis axis V suggests itself as the most likely, and a rotation through a negative angle of 21° 49′ seems suitable; in the second analysis axis IV may be taken, and the rotation made through a negative angle of 14° 1′.

In analysis *A* two factors, V and VI, appear at this stage to have something to do with reflective thinking. In locating the fifth factor we provisionally assumed that it might be treated as autistic thinking, expecting (a) that it would have heavy loads on variables 12, 28, and 70, all of which measure introspective behavior, and (b) that a variable such as 68, which deals with thinking in a narrower sense, free from introspective bias, would be relatively untouched by it. It turns out, however, that the loads of V and VI on variable 68 are about equal, being .429 and . 412. This result seems wrong. The probability, then, is that the more important factor in this variable is liking for thinking, rather than autistic thinking. This leads to the rotation of axes V and VI through a negative angle of 45°.

A further rotation commends itself. Axis A-VII has only four significant loads. Two of these are on variables dealing with athletics, 19 and 84, the other two measuring preference for practical rather than theoretic affairs, viz., 13 and 4. On the average these loads can be increased by rotating VII with IX through a positive angle of 20° 21′.

Results

The transformations giving effect to all these rotations are presented in Tables 4 and 5, and the factorial matrices to which they lead in Tables 6 and 7.

TABLE 4

I	II	III	IV	V	VI	VII	VIII	IX
–.3558	–.5157	–.3560	.2762	–.2354	.3721	.1618	–.2377	.3574
.7781	–.3951	–.2772	.2422	.1885	–.1335	.1234	.1738	.0641
.3104	.4773	–.0158	–.0665	.2923	.5993	.1136	–.3031	.3490
.1085	–.5006	.8018	–.1674	.0784	.2207	.0152	–.0690	.0829
.2519	.1787	.1656	–.0487	–.6396	–.3638	.3781	–.3436	.2694
–.2664	–.0486	.0266	.0256	.6379	–.5008	.3204	–.3113	.2613
–.0747	.2426	.3519	.8792	.0018	.0227	.0752	.1804	.0101
.0504	.0116	.0301	.0643	–.0326	–.2273	–.7738	–.0452	.5819
–.1312	.0786	.0228	–.2293	–.0233	.0177	.3093	.7532	.5096

TABLE 5

I	II	III	IV	V	VI
.0040	–.7832	.2820	.4285	.3468	–.0564
–.2917	–.0009	.5427	–.6796	.3982	–.0088
–.1960	–.5097	–.6569	–.4738	–.0626	–.2046
–.7836	.2725	–.1877	.3602	.2917	–.2486
–.4753	–.1997	.0934	.0197	–.4260	.7373
.1915	.1125	–.3880	–.0064	.6714	.5911

TABLE 6

1 Are you ordinarily a carefree individual?
2 Do you usually have difficulty in starting a conversation with strangers?
4 Do you prefer to read about a thing rather than to experience it?
6 Do you hesitate to lend your personal property even to close friends?
7 Are you inclined to be considerate of other people's feelings?
10 Are you relatively unconcerned about what others think of your actions?
12 Are you inclined to analyze the motives of others?
13 Do you consider yourself a practical individual rather than one who theorizes?
14. Do you usually keep in close touch with things going on around you?
15 Are you inclined to worry over possible misfortunes?
16 Do you often have the "blues"?
18 Are you inclined to keep in the background on social occasions?
19 Are you more interested in athletics than in intellectual things?
25 Would you rate yourself as an impulsive individual?
26 Do you enjoy getting acquainted with most people?
28 Do you frequently find yourself in a meditative state?
29 Are you inclined to be over-conscientious?
30 Do you often crave excitement?
32 Are you inclined to ponder over your past?
40 Are you inclined to stop and think things over before acting?
45 Are you less attentive than most individuals to things going on around you?
61 Do you like to discuss the more serious questions of life with your friends?
62 Do you like to try your wits in solving puzzles?
67 Would you rate yourself as a happy-go-lucky individual?
68 Do you enjoy thinking out complicated problems?
70 Are you inclined to be introspective, that is, to analyze yourself?
81 Are you usually unconcerned about the future?
84 Do you usually become completely absorbed in watching an athletic contest?
85 Can you relax yourself easily when sitting or lying down?
87 Are you more alert to your immediate surroundings than the average person?

TABLE 6

	I	II	III	IV	V	VI	VII	VIII	IX
1	.804	.058	.034	–.057	.257	–.250	–.082	–.020	–.123
2	–.079	.195	–.660	.026	–.087	–.054	.116	.041	.114
4	–.219	.105	–.286	–.006	.062	–.039	–.331	–.109	–.050
6	–.154	.034	–.095	–.051	–.091	.056	.001	–.140	.257
7	–.105	.107	.242	.124	.002	.337	.008	.008	.210
10	.350	.155	–.083	–.276	.118	–.008	.082	.293	.136
12	.023	–.152	.047	.120	–.384	.460	.092	.112	–.017
13	–.045	.233	.247	.008	–.014	–.181	.346	.024	.000
14	.057	.189	.487	.634	–.063	–.111	.048	.047	.073
15	–.344	–.377	–.257	.138	–.136	.142	–.034	–.258	.120
16	–.236	–.342	–.453	.179	–.237	.096	–.064	–.067	.256
18	–.184	.231	–.677	–.038	–.071	–.025	–.025	–.025	–.100
19	.219	–.178	–.124	.072	–.182	–.211	.403	.031	–.153
25	.328	–.404	.028	.032	.289	.128	–.154	.242	–.076
26	.078	–.101	.662	–.020	–.054	–.180	.085	.016	.004
28	–.025	–.235	–.255	.026	–.493	.414	–.132	–.104	.191
29	–.365	.033	–.199	.172	–.182	.139	–.020	.283	.097
30	.368	–.285	.056	.047	–.104	–.132	.156	–.138	.189
32	–.117	–.369	–.386	.111	–.098	.336	–.120	–.050	–.030
40	–.299	.460	.033	.110	–.421	.146	.103	–.343	–.111
45	–.037	–.239	–.550	–.570	–.066	.035	–.014	.003	–.029
61	–.043	–.103	.079	.084	–.397	.238	–.174	–.010	.055
62	.195	.288	.210	–.011	.201	.314	.165	–.179	.272
67	.807	–.058	–.034	.057	.198	–.206	.082	.020	.123
68	.045	.431	.040	.054	–.012	.595	.025	.018	.230
70	–.073	–.180	–.077	.051	–.496	.500	.036	–.008	–.172
81	.367	.170	–.058	–.023	.026	–.169	–.029	.131	–.088
84	.194	–.154	.176	.124	+.007	–.052	.442	.043	.105
85	.271	.294	.196	–.060	.171	.030	.181	–.319	–.226
87	–.041	–.020	.191	.552	.011	.131	–.062	–.043	–.101

TABLE 7

7 In a difficult or exacting situation have you often found yourself perspiring?
10 Do you feel compelled to change your bodily position frequently while sitting?
11 Do you like to change from one type of work to another frequently?
20 Are you easily disturbed by distracting stimuli while doing mental work?
24 Are you inclined to be slow and deliberate in movement?
31 Do you express such emotions as delight, sorrow, anger, and the like readily?
34 Are you easily startled by unexpected stimuli?
40 Are you inclined to stop and think things over first before acting?
41 Do you have any "nervous habits," like chewing your pencil or biting your finger nails?
43 Does it annoy you to see a person clean his finger nails in public?
44 Do you usually eat more rapidly than the average person even though you have plenty of time?
50 Are you inclined to rush from one activity to another without pausing for rest?

60 Would you like a position in which you change from one kind of task to another frequently during the day?
63 When listening to a lecture, does your hand keep active in writing or drawing when not taking notes?
67 Would you rate yourself as a happy-go-lucky individual?
69 Do you frequently suffer from insomnia?
72 Do you often find yourself hurrying to get places even when there is time?
74 Are you inclined to be quick in your actions?
79 Do you think you use up more energy than most individuals in getting things done?
83 Can you turn out a large amount of work in a short time?
85 Can you relax yourself easily when sitting or lying down?
89 Would you rate yourself as a talkative individual?
95 Do you dislike very much to be interrupted in a task which you want to finish?

TABLE 7

	I	II	III	IV	V	VI
7	–.193	–.244	.051	–.031	.221	–.251
10	–.090	–.416	.179	–.038	.238	–.364
11	.110	–.727	–.207	–.167	–.225	–.017
20	–.382	–.177	.282	.021	.311	.123
24	.016	.250	–.111	–.622	.043	.056
31	–.018	–.171	.258	.166	.116	–.159
34	–.025	–.141	.414	–.109	.280	.184
40	–.323	.132	–.053	–.338	–.158	–.076
41	–.003	–.271	.346	–.037	.156	–.095
43	–.017	.041	.376	.037	–.044	.239
44	.014	–.163	.010	.376	.207	–.231
50	–.064	–.162	–.079	.517	.226	.282
60	.095	–.717	–.231	–.112	–.223	–.065
63	–.026	–.277	.356	.135	.068	–.029
67	.661	.000	.000	.165	.000	.000
69	–.125	–.359	.218	–.178	.216	.252
72	–.283	–.310	–.172	.254	.219	–.245
74	–.175	–.189	–.017	.795	–.029	–.010
79	–.044	–.237	–.024	–.077	.383	–.021
83	–.118	–.208	.005	.565	–.235	.066
85	.190	.316	–.015	.082	–.454	–.279
89	.399	–.175	–.012	.312	.148	.002
95	–.320	–.114	.126	.002	.144	–.067

In Table 8 a comparison is made of the loads on the three common variables, 40, 67, and 85, of the corresponding factors in the two analyses. In the Table the analyses are called A and B, respectively, and the corresponding factors appear to be as follows: A-I corresponds to B-I; A-II corresponds to B-II; A-V corresponds to B-IV; A-VIII corresponds to B-V.

TABLE 8

	A	B	A	B	A	B	A	B
	I	I	II	II	V	IV	VIII	V
40	−.299	−.323	.460	.132	−.421	−.338	−.343	−.158
67	.807	.661	−.058	.000	.198	.165	.020	.000
85	.271	.190	.294	.316	.171	.082	−.319	−.454

The agreement between the two analyses with respect to these three variables could be made somewhat closer, but this would be done at the expense of the general intelligibility of the interpretation, and on the whole it seems better to leave it as it is given above.

Description of the Common Factors

We shall now proceed to consider the common factors which have been obtained by these two analyses, and we may begin with factors which are found in both. Little time need be spent over the first two of these, viz., Surgency and Persistence, for, if our procedure has been at all correct, they may be taken as identical in the two analyses. In addition to them, there appear to be two further factors present in both batteries, for A-V may be identified with B-IV, and A-VIII with B-V. The facts given above in Table 8 are a first indication of this, and a comparison of the factor loads of the other variables seems to confirm the hypothesis.

Let us look at the chief loads of A-V and B-IV. They can easily be understood as forming a single picture. Positively, the person who is heavily endowed with this factor will be quick in his actions (B 74), can do much work in a short time (B 83), will hustle between jobs (B 50), will bolt his food (B 44), will tend to be talkative (B 89), impulsive (A 25), carefree (A 1), and generally in a hurry (B 72). On the other hand and in consequence, he will not be slow and deliberate (B 24), he will not be introspective (A 70), or meditative (A 28), nor will he stop and think things over before acting (A 40 and B 40), he will not be prone to discuss the serious questions of life with his friends (A 61), nor to analyze their motives (A 12). The factor is, with reference to behavior as a whole, what fluency is with regard to speech. Perhaps we may call it Flexibility.

If, following the above procedure, we compare A-VIII with B-V in respect to the common variables, we find that the analyses are congruent: both have negative loads on 85 (relax easily) and on 40 (thinking before acting). Thus at the outset we get a picture of a person who does not relax readily and who tends to be active. To this A-VIII adds that he is impulsive (25), over-conscientious, (29) but that he does not worry (15), and is relatively unconcerned about what others think of him (10). From B-V we learn that he tends to waste

energy (79), to hustle between jobs (50), to hurry even when he is in time (72), that he cannot sit still (10), that he is easily disturbed (20) and easily startled, (34), but doesn't adapt himself easily to new work (11 and 60), and in spite of his hustle and hurry has not a great output under pressure (83). Finally, he suffers from a slight tendency to insomnia (69). The name Tension, already suggested, appears to be satisfactory.

If Tension is compared with the previous factor Flexibility, it will be noted that there are certain forms of behavior to which they both contribute. Both lead to hustling between jobs, to hurrying even when in time, to impulsive action, and neither leads to planning before action. But at the same time there are notable differences. Tension makes for over-conscientiousness, Flexibility does not. It also contributes to waste of energy, to the inability to sit still, to insomnia, in a way which Flexibility does not. The tense person is readily disturbed and startled, and does not relax easily. The flexible person is normal in these respects.

From the first analysis we have five other factors. The first of these, A-III, may be called Sociability, its keynote being 26 (do you enjoy getting acquainted with most people?), with a load of .66. The person who has this quality has no difficulty in starting a conversation with strangers (2), comes well forward on social occasions (18), is in touch with things going on around him (14 and 45) and is not readily subject to the "blues" (16). Read from the opposite end, as the Guilfords take it, it may be called Shyness. Factor A-IV has three definitely significant loads, namely, 14 (Do you usually keep in close touch with things going on around you?), 45 (are you less attentive than most individuals to things going on around you?), 87 (are you more alert to your immediate surroundings than the average person?). These three may be taken as the same question in slightly different form, and the appearance of the factor, which may be called Alertness, is due almost entirely to this repetition. Only one other load, that of 10, is probably significant, and the concern for what others think of one's actions, to which it refers, is a natural consequence of alertness to one's environment.

The next factor, A-VI, leads to an interest in complicated problems (68) and in problems generally (62). It tends to make a man introspective (70) and meditative (28), leads him to analyze the motives of others (12), and to ponder over the past (32); it keeps him from being wholly carefree (1) and happy-go-lucky (67), encourages him to discuss the serious questions of life with his friends (61) and gives him some preference for intellectual things rather than athletics (19). The Guilfords call one of their factors Liking for Thinking. It

has relatively heavy loads on 68 and 62, the two variables concerned with the solving of problems, but rather strangely it involves a power to relax easily (85), makes one considerate to others (7) and may even favor a happy-go-lucky disposition (67). The title of the factor does not seem in keeping with the contents, and may more appropriately be applied to the factor as it appears in our analysis.

A-VII has only four significant loads. The person who has this quality gets easily absorbed in an athletic contest (84), prefers athletic affairs to the things of the mind (19), regards himself as a practical man rather than a theorizer (13) and would rather do things than read about them (4). The quality may be called Interest in Action.

The last factor in the first analysis, A-IX, has six loads exceeding .2, but none reaching .28. They do not fall into any clearly defined pattern and we make no attempt to identify the factor: as given above it probably contains a large amount of error.

There remain to be considered two factors in the second analysis.

B-III presents the following features. It makes one easily startled (34) and easily distracted (20). Its possessor is annoyed by seeing someone clean his finger nails in public (43); he is inclined to "doodle" (63), to have nervous habits (41), and to express his emotions easily; he is made uncomfortable by changes of work (60 and 11) and tends to suffer from insomnia (69). The factor has some resemblance to the Guilfords' first factor, N, and may be given the same name, Nervousness.

There are no very high loads in B-VI, but eight of the variables have coefficients which are probably significant. The possessor of this quality does not wriggle (10), does not perspire easily in exacting situations (7), does not relax easily (85), does not gobble his food (44), does not hurry unnecessarily (72) but does not waste time between jobs (50), does not sleep too well (69) and does not like to see people clean their finger nails in public (43). The quality names itself: Inhibition.

This completes our analysis. The factors which we have mentioned are not all put forward with the same degree of assurance. Some of them have appeared elsewhere, and may be regarded as fairly well established, while others appearing for the first time require confirmation.

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A COEFFICIENT OF IMBALANCE FOR CONTENT ANALYSIS*

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This article presents a Coefficient of Imbalance applicable to any type of communication that may be classified into favorable content, unfavorable content, neutral content, and non-relevant content. The combined influence of the average presentation of relevant content and the average presentation of total content is reduced to two components, the coefficients of favorable imbalance and of unfavorable imbalance. A precise definition of imbalance is developed and measured against ten criteria.

I. *Introduction*

An important aspect of research on interpersonal relationships is the precise statement of relationships between (a) the symbols found in communications and (b) various other types of behavior. The development of personality analysis and propaganda analysis has been brought about primarily by investigating and inventing situations in which rich interpersonal communications and other behaviors occur together spontaneously or are elicited. The insights of Freud, for example, arose from his intensive study of the symbolic communications of his neurotic patients under the special conditions of the psychoanalytic interview. The psychoanalysts, however, have rarely made explicit the operations involved in drawing conclusions from interview data. Perhaps because the data secured by the free association technique were so immediately suggestive, Freud and his followers have contented themselves with the creation and elaboration of fruitful but ambiguous hypotheses. They have made little advance in techniques of communications analysis necessary for specifying the conditions under which their theories might be confirmed.

Analysis of the spontaneous verbalizations of children in play situations, especially the work of Piaget, has also proved successful in the study of the development of personality. Although the majority of such studies tend to rely upon impressionistic analysis, some investigators of personality structure, such as Rorschach and his followers, have developed precise methods for classifying the symbol data which are elicited in standardized stimulus situations.

* The authors wish to express their appreciation to Mr. Harold Elsten, Dr. Natan C. Leites, and Mr. Ithiel Pool for valuable suggestions and criticisms.

Like personality analysts, the investigators in the recently developed field of propaganda and public opinion have tended to rely to a large extent on impressionistic descriptions of communications and of the situations in which communications take place. It is true that these investigators have devised fairly satisfactory quantitative methods for the study of attitudes.* Questionnaires and public opinion polls facilitate quantitative analysis of simple attitudes by restricting the verbal responses of subjects to a choice of one of a few words (e.g., "agree," "disagree," "no opinion"). However, when concerned with mass communications—the press, radio, movies, speeches, etc.—the method of analysis has tended to remain largely impressionistic.

Impressionistic judgments suffice for broad classification of symbol data and description of gross temporal changes in the content of mass communications. Thus, we may report reliably that a certain movie is manifestly anti-Nazi, or that the contents of a certain newspaper changed from pro-isolation to pro-intervention. But if we wish to develop precise hypotheses concerning mass communications, there is a need for quantitative analysis of symbols.

In recent years a number of studies have been reported which are of methodological interest in so far as they have employed quantitative content analysis. In general, the method consists of tabulating the occurrences of content units (subject-predicate, assertion, or group of sentences) and classifying them into such categories as favorable, unfavorable, or neutral presentation.**

The further development of quantitative content analysis requires the development of (a) fruitful and reliable categories for classification of content units, and (b) quantitative methods of expressing the total picture provided by the classified data: i.e., the development of general formulae to serve as definitions of general concepts.

The elaboration of categories has already made considerable pro-

* For a summary of the outstanding attitude-scales, such as those of L. L. Thurstone and of R. Likert, see Gardner and Lois Barclay Murphy and Theodore M. Newcomb, *Experimental social psychology*, New York: Harpers, 1937, Ch. 13.

** Some of the pioneering studies are the following: Gordon W. Allport and Janet M. Faden, The psychology of newspapers: five tentative laws, *Public Opinion Quarterly*, 1940, 4, 687-703; Thomas S. Green, Jr., Mr. Cameron and the Ford Hour, *Public Opinion Quarterly*, 1939, 3, 669-75; Harold D. Lasswell, The World attention survey, *Public Opinion Quarterly*, 1941, 5, 456-462; David Nelson Rowe, Japanese propaganda in North China, 1937-1938, *Public Opinion Quarterly*, 1939, 3, 564-80; Douglas Waples and Bernard Berelson, *Public Communications and Public Opinion*, Chicago: Univ. Chicago Press, 1941; Quincy Wright and C. J. Nelson, American attitudes toward Japan and China, 1937-1938, *Public Opinion Quarterly*, 1939, 3, 46-62. For a discussion of the methodological problems of content analysis, see N. C. Leites and I. de Sola Pool, On content analysis, Document No. 26, Experimental Division for the Study of War Time Communications, Library of Congress.

gress, especially as a result of the contributions of Dr. Harold D. Lasswell† and Dr. Natan C. Leites.‡

It is the purpose of this paper to develop a general formula which may be applied to classified content data in order to present an overall estimate of the degree of imbalance; i.e., the extent to which favorable, neutral, or unfavorable treatment is accorded to the topic or symbol under analysis. This Coefficient of Imbalance is intended to be applicable to all types of communications—including mass communications, psychoanalytic interviews, conversations, literary works, cartoons, pictures, etc.—except those in which the communication is arbitrarily restricted to specified symbols, as in multiple-choice or yes-no answers to questionnaires.

Moreover, this Coefficient is intended to apply to any trait for which a communication is analyzed, provided that it is possible to classify units of content according to the occurrence of the trait, occurrence of the opposite trait, and non-occurrence of the trait. Thus, while the discussion below is in terms of "Favorable" and "Unfavorable" contents, the meaning need not be restricted to pro- and anti-references to a symbol.

It is necessary to have a set of defined terms by which we may refer to those aspects of communications with which we are concerned. A "unit of content" is defined as the unit of meaning selected for analysis. The unit employed in a particular content analysis will depend upon the medium of the communication and the trait under consideration. In analyses of verbal material, the most frequently employed units of content are functional groups of sentences (e.g., newspaper articles), individual sentences, subject-predicate phrases, assertions, and terms.

To designate the entire content of a communication we shall use the term "the total content," which may be expressed quantitatively as the total number of units of content. By "relevant content" we mean the total number of units of content which contain the trait under investigation. The "total content" includes both "relevant content" and "non-relevant content" (those units of content in which the trait does not occur). The "relevant content" includes the "favorable content" (those units of content which contain favorable occur-

† Harold D. Lasswell, A Provisional Classification of Symbol Data, *Psychiatry*, 1938, 1, 197-204;—and Associates, The politically significant content of the press: coding procedures, *Journalism Quarterly*, 1942, 19:1, 12-23;—Communications research and politics, in *Print, Radio, and Film in a Democracy* (edited by Douglas Waples), Chicago: Univ. Chicago Press, 1942, pp. 118-32;—Analyzing the content of mass communication: a brief introduction, Document No. 11, Experimental Division for the Study of War Time Communications, Library of Congress.

‡ Waples and Berelson, *ibid.* Appendix A; and unpublished manuscripts by N. C. Leites and I. de Sola Pool.

rences), the "unfavorable content" (those units of content which contain unfavorable occurrences) and the "neutral content" (those units of content which contain occurrences of the trait which are neither favorable nor unfavorable).

To illustrate the use of these terms, let us say that we are investigating the bias in the presentation of Germany in a political pamphlet. We select an assertion as our unit of content. The total content is the total number of assertions on all topics in the pamphlet. (If the pamphlet is long, an estimate of the total content may be made by some short method such as determining the number of assertions on sample pages and multiplying the average per page by the total number of pages.) Only assertions referring to Germany are relevant content and each of these is recorded as either favorable, unfavorable, or neutral. After obtaining the frequency of content units which fall into each of these categories, it is desirable to express the total picture by means of a single numerical value, a Coefficient of Imbalance.

II. *Development of the Coefficient of Imbalance.*

A satisfactory Coefficient of Imbalance must meet the requirements set up by a definition of "imbalance" which is consistent with current usage. Such a definition, making explicit the concept of imbalance in communications, is provided by the following ten criteria. In each criterion, the variables not specified as changing are understood to be held constant, with the exception of non-relevant content.* Detailed descriptions of the criteria are to be found in Section III below.

1. The Coefficient should always increase in the positive direction when the frequency of units of favorable content increases.
2. The Coefficient should always increase in the negative direction when the frequency of units of unfavorable content increases.
3. The Coefficient should always decrease in absolute value when the frequency of units of neutral content increases.
4. The Coefficient should always decrease in absolute value when the number of units of total content increases.
5. If there is no relevant content, the Coefficient must be zero.
6. If all the units of relevant content are neutral, the Coefficient must be zero.
7. If the number of units of favorable content is equal to

* Cf. p. 117.

the number of units of unfavorable content, the Coefficient must be zero.

8. If the number of units of favorable and unfavorable content are not equal, the Coefficient must not be zero.
9. If all the relevant content is favorable (or unfavorable), any variation in the frequency of units of favorable (or unfavorable) content should provide a directly proportionate variation in the Coefficient.
10. If there is no neutral content, the Coefficient must vary directly as the ratio of the favorable to the unfavorable, whenever the difference between favorable and unfavorable content remains constant.

In developing the Coefficient of Imbalance, let us consider first an analysis of the relevant content. The relevant content may be classified into any number of sub-categories which may be regarded as falling into the main categories: favorable, unfavorable, and neutral content. Analysis into different degrees of favorableness and unfavorableness will determine the number (m) of sub-categories. It is theoretically possible to assign a numerical value to each sub-category (x_i).

Considering the frequency of content units which are in each sub-category as a weight (w_i), we may obtain a weighted average presentation of the relevant content (A); i.e.,

$$A = \frac{x_1 w_1 + x_2 w_2 + x_3 w_3 + \cdots + x_m w_m}{w_1 + w_2 + w_3 + \cdots + w_m} = \frac{\sum_{i=1}^{m} x_i w_i}{\sum_{i=1}^{m} w_i}. \tag{1}$$

Let us consider next an analysis of the total content. The total content includes all of the sub-categories of the relevant content, and in addition it includes any number (M) of non-relevant sub-categories (N_j). Again, each sub-category of the total content (x_i and N_j) theoretically may be assigned a numerical value and the frequencies of content units in each sub-category (w_j) may be regarded as weights so that a weighted average presentation of the total content (T) is obtainable; i.e.,

$$T = \frac{\sum_{i=1}^{m} x_i w_i + \sum_{j=1}^{M} N_j w_j}{\sum_{i=1}^{m} w_i + \sum_{j=1}^{M} w_j}. \tag{2}$$

In assigning values to the sub-categories of the total content, (x_i and N_j), it is desirable to indicate favorable sub-categories as having a positive direction. Hence all values attached to favorable sub-categories will be positive. Since unfavorable content is the opposite of favorable, it follows that unfavorable sub-categories should be assigned negative values. If there are several sub-categories of favorable content and of unfavorable content, the numerical values assigned to them must be determined by empirical evaluation of the degree or intensity of favorableness or unfavorableness. In the absence of a means of differentiating between degrees of favorableness or unfavorableness, all favorable units of content may be placed in a single category with the value of plus one (+1) and all unfavorable units of content may be placed in a single category with the value of minus one (−1).

The neutral category by definition contains relevant content units which are neither favorable nor unfavorable but lie midway between them and so must be assigned the value zero (0). Similarly, non-relevant units of content are neither favorable nor unfavorable and so must also be assigned the value zero. Hence, non-relevant units of content may be considered as being in a single category (which makes $M = 1$) and the weight of this category (w_j) is simply the number of non-relevant units of content, which we may represent by N without a subscript, $w_j = N$. (3)

Let the favorable content be represented by "y" and the unfavorable content by "v" so that

$$y = x > 0 \tag{4}$$

$$-v = x < 0 . \tag{5}$$

Making these substitutions in the weighted average presentation of relevant content (1) and in the weighted average presentation of the total content (2) and making the appropriate changes in the weight subscripts, we obtain

$$A = \frac{\sum_{i=1}^{m_y} y_i \, w_{y_i} - \sum_{i=1}^{m_v} v_i \, w_{v_i}}{\sum_{1}^{m_y} w_{y_i} + \sum_{1}^{m_v} w_{v_i} + n} \tag{6}$$

(where n = frequency of units in the neutral category) or

$$A = \frac{\sum_{i=1}^{m_y} y_i \, w_{y_i} - \sum_{i=1}^{m_v} v_i \, w_{v_i}}{r} \tag{7}$$

(where r = total units of relevant content). Also since $N_j = 0$ and $\sum w_j = N$, we obtain from (2)

$$T = \frac{\sum_{i=1}^{m_y} y_i\, w_{y_i} - \sum_{i=1}^{m_v} v_i\, w_{v_i}}{\sum_{i=1}^{m_y} w_{y_i} + \sum_{i=1}^{m_v} w_{v_i} + n + N} \tag{8}$$

$$= \frac{\sum_{i=1}^{m_y} y_i\, w_{y_i} - \sum_{i=1}^{m_v} v_i\, w_{v_i}}{t} \tag{9}$$

(where t = number of units of total content).

To simplify the notation, let $f = \sum_{i=1}^{m_y} y_i\, w_{y_i}$, and $u = \sum_{i=1}^{m_v} v_i\, w_{v_i}$ so that $A = \frac{f-u}{r}$ and $T = \frac{f-u}{t}$.

(It will be noted that $r = f + u + n$ and that $t = f + u + n + N$, if $m_y = m_v = 1$).

Neither the average of relevant content (A), nor the average of total content (T), will alone serve as an adequate coefficient of imbalance. The former meets all the criteria except 4 and 9; the latter meets all except 3* and 10. We may expect that some combination of these two averages will meet all of the criteria, for we observe that where one fails the other is successful.

The mathematically inclined reader may demonstrate for himself that none of the combinations resulting from the four fundamental operations of mathematics (addition, subtraction, multiplication, and division**) will give a formula which will satisfy all ten of the criteria. Nor are the arithmetic, geometric, or harmonic means satisfactory. However, separate formulae for favorable and unfavorable imbalance may be necessary. Thus, one of these procedures may provide a formula which can be broken down into two components: a coefficient of favorable imbalance and a coefficient of unfavorable imbalance. Such a result is obtained by analysis of the product of the two averages.

* Since "t" is required to be held constant. an increase of neutral content implies an equal decrease in non-relevant content so that the value of "t" remains unchanged. Cf. p. 117 for a fuller discussion of "t".

** Division provides an index to the attention devoted to the topic or symbol, $\frac{r}{t}$.

$$AT = \frac{f-u}{r} \cdot \frac{f-u}{t} \tag{10}$$

$$= \frac{f^2 - fu}{rt} - \frac{fu - u^2}{rt}. \tag{11}$$

Let the first component be called the Coefficient of favorable imbalance, C_f, and the second be called the Coefficient of unfavorable imbalance, C_u. Thus, the combined influence, AT, may be regarded as the difference between the Coefficients of favorable and unfavorable imbalance.

$$AT = C_f - C_u, \tag{12}$$

where

$$C_f = \frac{f^2 - fu}{rt} = \frac{f-u}{r} \cdot \frac{f}{t} \tag{13}$$

and

$$C_u = \frac{uf - u^2}{rt} = \frac{f-u}{r} \cdot \frac{u}{t}. \tag{14}$$

It may be observed that the factors making up these Coefficients may be given a rational interpretation in terms of quantitative concepts. The first factor in each, $\frac{f-u}{r}$, is the weighted average presentations of relevant content (A). The second factor, $\frac{f}{t}$ or $\frac{u}{t}$, is the relative frequency of the favorable or unfavorable content. Hence, the second factor represents the extent to which the total content was utilized to present favorable (or unfavorable) content. When the imbalance is favorable, i.e., $f > u$, it is the extent to which the total content was utilized to present favorable content that is the determining factor. Similarly, when the imbalance is unfavorable, i.e., $f < u$, the determining factor is the extent to which the total content was untilized to present unfavorable content. Hence, when $f > u$ the imbalance is favorable and the Coefficient is given by C_f; when $f < u$ the imbalance is unfavorable and the Coefficient is given by C_u. These two formulae may be used as a Coefficient of Imbalance, which expresses quantitatively the imbalance of any communication with respect to the topic or symbol under investigation.

The primary advantage of such a quantitative measure is that hypotheses involving imbalance may be tested by employing statistical techniques. Thus a hypothesis concerning the difference in imbal-

lance between two groups of communications may be confirmed or disconfirmed by testing the significance of the difference between the means of the Coefficients of Imbalance for the two groups. Similarly, the Coefficients of Imbalance for a given symbol in a group of communications may be correlated with (a) the Coefficients for another symbol in the same group, (b) the Coefficients for a related group of communications, or (c) another set of quantitative measures of some non-symbolic event (such as election returns or income of the author).

III. *Demonstration of the Adequacy of the Coefficient of Imbalance.*

Criterion 1. *The Coefficient should always increase in the positive direction when the frequency of units of favorable content increases.*

Let df represent a positive increment in the favorable frequency, f. Then, proving that Criterion 1 is satisfied by the formulae requires proving the inequalities:

$$\frac{(f+df)^2 - u(f+df)}{(r+df)t} > \frac{f^2 - fu}{rt} \quad \text{when } f > u$$

and

$$\frac{u(f+df) - u^2}{(r+df)t} > \frac{uf - u^2}{rt} \quad \text{when } f < u.$$

This demonstration may be made algebraically or by using the calculus and differentiating C_f and C_u with respect to f and showing that the slope does not change sign or become zero for positive values of f.

Criterion 2. *The Coefficient should always increase in the negative direction when the frequency of units of unfavorable content increases.*

Let du represent a positive increment in the unfavorable frequency. Then Criterion 2 becomes:

$$\frac{f^2 - (u+du)f}{(r+du)t} < \frac{f^2 - uf}{rt} \quad \text{when } f > u$$

and

$$\frac{f(u+du) - (u+du)^2}{(r+du)t} < \frac{fu - u^2}{rt} \quad \text{when } f < u.$$

This demonstration may also be made algebraically or by using the calculus and differentiating with respect to the independent variable.

Criterion 3. *The Coefficient should always decrease in absolute value when the frequency of units of neutral content increases.*

Since a unit of neutral content is defined as a unit of relevant content which is neither favorable nor unfavorable, but the midpoint between these two, this requirement is essential to making the coefficient consistent with the definition of neutrality.

Let dn represent a positive increment in neutral frequency and this becomes, for $f > u$,

$$\frac{f^2 - fu}{(r + dn)t} < \frac{f^2 - fu}{rt} \quad \text{or} \quad r + dn > r$$

and for $u > f$:

$$\frac{uf - u^2}{(r + dn)t} < \frac{uf - u^2}{rt} \quad \text{or} \quad r + dn > r.$$

Criterion 4. *The Coefficient should always decrease in absolute value when the number of units of total content increases.*

This requirement means that a communication which uses only a small proportion of its available units of content to present a given number of units of relevant content is less imbalanced than one which uses a large proportion of its available units of content to present the same number of units of relevant content. In other words, if the total units of content increase, the number of units in each component of the relevant content must increase in the same proportion in order that the same degree of imbalance be maintained.

Let dt represent a positive increment in the number of units of total content and the proof is exactly the same as for Criterion 3.

Criterion 5. *If there is no relevant content, the Coefficient must be zero.*

If only non-relevant units of content occur, the communication cannot be imbalanced with respect to the topic or symbol under analysis.

When $f = u = r = 0$,

$$C_f = \frac{f^2 - fu}{rt} = \frac{0}{0}, \text{ which is indeterminate.}$$

With f as the only independent variable,

$$\lim_{f \to 0} \frac{f^2 - fu}{ft} = \lim_{f \to 0} \frac{2f}{t} = 0$$

by differentiation of the numerator and denominator with respect to f.

Also, $C_u = \dfrac{fu - u^2}{rt} = \dfrac{0}{0}$, which is indeterminate.

With u as the independent variable,

$$\lim_{u \to 0} \frac{uf - u^2}{ut} = \lim_{u \to 0} \frac{-2u}{t} = 0.$$

Criterion 6. *If all the units of relevant content are neutral, the Coefficient must be zero.*

If only neutral units occur, the communication cannot be imbalanced.

If

$$f = u = 0, r = n,$$

then

$$C_f = \frac{f^2 - fu}{rt} = 0 \text{ and } C_u = \frac{uf - u^2}{rt} = 0.$$

Criterion 7. *If the number of units of favorable content is equal to the number of units of unfavorable content, the Coefficient must be zero.*

In this case, the communication is exactly balanced.

If

$$f = u,$$

then

$$C_f = \frac{f^2 - fu}{rt} = 0 \text{ and } C_u = \frac{uf - u^2}{rt} = 0.$$

Criterion 8. *If the number of units of favorable and unfavorable content are not equal, the Coefficient must not be zero.*

In this case, the communication must be imbalanced.

When

$$f > u, f^2 > fu \text{ and } f^2 - fu \neq 0.$$

Therefore

$$C_f = \frac{f^2 - fu}{rt} \neq 0.$$

Similarly when

$$u > f, C_u = \frac{uf - u^2}{rt} \neq 0.$$

Criterion 9. *If all of the relevant content is favorable (or unfavorable), any variation in the favorable (or unfavorable) content should provide a directly proportionate variation in the Coefficient.*

Consider two communications with the same total content, in each of which all of the relevant content is imbalanced in the same direction. If, for example, one publication has twice as many favorable content units as another for the same total content, a rational judgment indicates that the one is twice as imbalanced as the other. (Naturally, cases with unfavorable or neutral content as well as favorable content would make an exact rational comparison of two cases impossible.)

When

$$f = r\,,\ u = n = 0\,,\ \text{and } t \text{ is constant;}$$

$$C_f = \frac{f^2 - fu}{rt} = \frac{f}{t} \text{ and } C_u = \frac{uf - u^2}{rt} = 0.$$

Let a be the general factor for relative proportions of favorable content. If one communication with a Coefficient C'_f has "f" units of content, and the other with a Coefficient C''_f has "af" units,

$$C'_f = \frac{f}{t} \text{ and } C''_f = \frac{af}{t} = aC'_f.$$

Similar reasoning demonstrates that the Coefficient satisfies the criterion when $u = r$.*

Criterion 10. *If there is no neutral content, the Coefficient must vary directly as the ratio of the favorable to the unfavorable, whenever the difference between favorable and unfavorable content remains constant.***

(Thus a communication having no neutral, 6 favorable and 1 unfavorable references is more imbalanced than one having 60 favorable and 55 unfavorable—the former $\frac{f}{u}$ ratio being 6:1, the latter 12:11.)

* It is also true that increasing the total content by the factor "a" decreases the Coefficient by the factor "a" because "t" appears as a single factor in the denominator.

** When there is neutral content ($n \neq 0$), the Coefficient varies directly as the ratio of the favorable to the unfavorable-plus-neutral content $\left(\frac{f}{u+n}\right)$. This is a reasonable relationship since $u + n$ constitute the "non-favorable" relevant content, $(r - f)$, corresponding to the "non-favorable" or unfavorable content in the simpler case stated by the criterion.

Let "d" represent a positive increment to both the favorable and unfavorable frequency.

If $f > u$, the ratio f/u decreases,

$$\text{i.e.,} \quad \frac{f+d}{u+d} < \frac{f}{u}.$$

In this case the criterion states that

$$\frac{f+d-(u+d)}{f+d+u+d} \cdot \frac{f+d}{t} < \frac{f-u}{f+u} \cdot \frac{f}{t},$$

which may be proved.

If $u > f$, the formula for C_u is used and the inequality is also true.

The fact that there is a transition from one formula to the other requires a careful scrutiny of the functions at the point of juncture, $f = u$. The continuity of the Coefficient functions and their derivatives with respect to the variables f and u at the point $f = u$ must be demonstrated before the transition is justifiable.

Continuity of the function itself and its derivatives is questionable only when f or u is the independent variable. If n or t be the independent variable, there is no change from one formula to the other. The matter of considering t as a constant merits some further consideration inasmuch as t is the total units of content in the communication and so is the sum of the units of relevant content and non-relevant content; i.e., $t = f + u + n + N$. Hence variation in f (or any of the other components) would cause a variation in t. A more useful result is obtained by considering N as a balancing component when other variables change. An increase, for example, in f necessitates a corresponding decrease in N, and t would remain constant. If t is to be the only independent variable, only N could change since f, u, and n are to be constant.

Consider first the continuity of the function when f is the independent variable. Inspection of the formulae will reveal the continuity if all variables, except C and f, are held fast. As f approaches u through values less than u, C_u approaches 0, passes through 0, and as f exceeds u, C_f exceeds 0 by continually larger values. That is,

$$\lim_{f \to u} C = C \Big|_{f=u} = 0.$$

which satisfies the definition of continuity. Similar reasoning demonstrates the continuity of $C = \phi(u)$ when all independent variables except u are held fast.

Continuity of the first derivative may be demonstrated as follows: (Let ∂ be the symbol for partial differentiation.)

Case 1. $C = \phi(f, u, n, t)$. Let f be the independent variable.

$$C_f = \frac{f^2 - fu}{(f+u+n)t} \text{ for } f > u \quad \text{and} \quad C_u = \frac{uf - u^2}{(f+u+n)t} \text{ for } f < u.$$

$$\frac{\partial C_f}{\partial f} = \frac{(f+u+n)t(2f-u) - (f^2-fu)t}{(f+u+n)^2 t^2} = \frac{(f+u+n)(2f-u) - (f^2 - fu)}{(f+u+n)^2 t}$$

$$\frac{\partial C_u}{\partial f} = \frac{(f+u+n)t(u) - (uf-u^2)t}{(f+u+n)^2 t^2} = \frac{(f+u+n)u - (uf-u^2)}{(f+u+n)^2 t}$$

At the point $f = u$,

$$\left.\frac{\partial C_f}{\partial f}\right|_{f=u} = \frac{(2u+n)u - 0}{(2u+n)^2 t} = \frac{u}{rt}$$

At the point $f = u$,

$$\left.\frac{\partial C_u}{\partial f}\right|_{f=u} = \frac{(2u+n)u - 0}{(2u+n)^2 t} = \frac{u}{rt}$$

since $r = f + u + n$,

or more correctly expressed

$$\lim_{\substack{f \to u \\ f > u}} \frac{\partial C_f}{\partial f} = \frac{u}{rt} \quad \text{and} \quad \lim_{\substack{f \to u \\ f < u}} \frac{\partial C_u}{\partial f} = \frac{u}{rt}.$$

Hence, the first derivatives are the same as $f = u$.

Case 2. Let u be the independent variable.

$$\frac{\partial C_f}{\partial u} = -\frac{\partial C_u}{\partial f} \quad \text{and} \quad \frac{\partial C_u}{\partial u} = -\frac{\partial C_f}{\partial f}.$$

This equality becomes apparent if for u we substitute $-f$ and for f we substitute $-u$. The formulae are symmetrical with respect to f and u except for sign.

Demonstration of the continuity of the higher derivatives would be a more complex problem in the calculus. But the fact that the transition through the critical point, $f = u$, is made in a continuous fashion by the function and that the slope is continuous and equal for both formulae, at this point, is sufficient to insure against rejection of the formula on the grounds of discontinuity.

If the Coefficient is graphed with each of the four variables in turn being considered as the independent variable, it will be observed

that the function is continuous as the formulae are interchanged. The relatively constant and low degree of curvature with respect to each variable throughout the range of values to be considered may also be observed, indicating that there are no abnormal ranges in which a small change in one variable results in a much larger change in the Coefficient than occurs in any other equal range.

IV. *Summary.*

A Coefficient of Imbalance was developed which may be applied to any type of communication provided that it may be classified into the following categories: favorable content (f), unfavorable content (u), neutral content (n), and non-relevant content (N). The average presentation (A) of relevant content (r) and the average presentation (T) of total content (t) were considered as two main factors involved in the measurement of imbalance. It was shown that the combined influence of these two factors could be reduced to two components, the Coefficient of favorable imbalance and the Coefficient of unfavorable imbalance. These Coefficients are meaningful in that they represent: (a) the average presentation of relevant content $\frac{(f-u)}{r}$, and (b) the degree to which the opportunity to present the predominant direction was used $\frac{(f \text{ or } u)}{t}$. The formulae provide a precise definition of imbalance:

$$C_f = \frac{f^2 - fu}{rt} \quad \text{where} \quad f > u$$

and

$$C_u = \frac{fu - u^2}{rt} \quad \text{where} \quad f < u.$$

This Coefficient of Imbalance was shown to be adequate with respect to ten criteria which together express the concept of imbalance.

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FACTORIAL ANALYSIS OF THURSTONE'S SEVEN PRIMARY ABILITIES

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Intercorrelation coefficients among Thurstone's seven primary mental abilities scores were obtained from scores of 170 freshmen engineering students on the experimental edition of the *Primary Mental Abilities Tests.* These correlation coefficients were factored to four factors, interpreted as a general factor, a reasoning factor, a verbal factor, and, tentatively, a specific memory factor. The finding of a general factor for a college population corroborates the Thurstones' finding of a general factor for eighth-grade children.

Introduction

Considerable discussion has taken place among the proponents of factor analysis concerning the interrelationship of factors. The question has led to a clear division of thought on the matter with the following views being taken. Harman and Holzinger (5), Hotelling (6), Kelley (7, 8), and Spearman (12) have all contended that the correlations between the factors isolated should be zero. The factors isolated by the Hotelling-Kelley, and the Spearman-Holzinger methods are uncorrelated with each other. On the other hand, Tryon (20), Price (10), and Thompson (14) appear to agree with Thurstone (17) that it is not essential for factor theory to have to assume that factors must be entirely uncorrelated. The methods used by Thurstone and Tryon will produce either correlated or uncorrelated factors, the criterion being mainly dependent on which solution offers the most meaningful interpretation.

Thurstone (15), writing about the relationship between the seven factors of perception, number, space, verbal, memory, induction, and reasoning, which he had isolated in his study of a battery of 56 tests, states:

> The primary mental abilities are uncorrelated, or nearly so, as far as can be determined from the three studies so far completed. However, the composite scores in the present test battery will not be uncorrelated. One should expect these composite scores to have low positive correlations not exceeding .30 or .40. Most of the correlations will be lower. The reason for this circumstance lies in the compromise which was made in adapting the tests for practical use . . . . as a practical compromise we have selected for each primary factor two or three tests that are heavily saturated with it . . . . but since the separate tests are not pure tests of any factor, it is clear that the combination of two or three tests will

also be an impure measure of the factor. Since any pair of the composite scores has some factor in common, the correlations between them will be positive and low.

Recent studies reporting on the relationship of Thurstone's primary mental abilities with each other have shown that few of the factors correlate zero with each other, while others have been found to reach as high as .57. Bernreuter and Goodman (1) found, after they had correlated the Thurstone primary mental abilities, that the highest correlation was .45 between induction and reasoning. Moffie (9) found among the correlations he had obtained between the primary abilities r's as high as .49 and .56. Harrell and Faubion (4), correlating four of the primary abilities, found correlations of .41 between the space ability and reasoning, and a correlation of .54 between reasoning and induction. Of the ten correlations reported by Harrell and Faubion (4), six of them were above .30. The intercorrelations reported by Shanner and Kuder (11) for the primary abilities also show correlations that are fairly high. Between the factors of induction and reasoning they report an r of .54, space and induction correlate .49, while space and reasoning correlate .47. Ellison and Edgerton (2) report correlations of .54 between space and perception, .47 between induction and perception, .43 between number and verbal, and .42 between number and reasoning. Of the twenty-one intercorrelations reported by these writers, ten of them are above .30. Stalnaker (13) reports that eight of the correlations found for a group of male freshmen college students are above .30, six above .40, and one above .50. The highest correlation for these boys was .57 between induction and reasoning. The same writer, reporting on the intercorrelation for a group of female freshmen students, shows the highest correlation to be .56 between induction and reasoning. Of the twenty-one intercorrelations for these female freshmen students, nine are above .30, five are above .40, and one above .50. Thurstone (19), writing about the intercorrelations of the primary abilities for children, states: "These correlations are higher than the correlations between the primaries for adults." However, she gives no figures to show the actual size of these correlations.

It would appear from the above findings that these intercorrelations would not readily be acceptable to those who contend that there should be no relationship between factors. Nor are they quite in harmony with Thurstone's contention that the correlations should be positive but low, not exceeding .30 or .40.

Problem and Procedure

Taking cognizance of these intercorrelations, the writer raised

the question whether it would be possible to determine what factor, or factors, might account for the overlapping that has been reported in the literature to date. The writer then decided to resort to the technique of factor analysis in an effort to answer this question.

The intercorrelations used in the factorial matrix were computed from the scores made by 170 freshmen male engineering students who entered The Pennsylvania State College in 1938. These students volunteered to take the 16 Thurstone *Primary Mental Abilities Tests* which were constructed to measure the seven abilities, upon being offered a psychograph of their own abilities based upon the test scores they obtained on these examinations. The mean age of the group was 19.58 years.

Some concern was felt about the influence this method of obtaining subjects would have on the representativeness of the group as a whole. However, a preliminary investigation (3) disclosed that the students thus obtained were a good sampling of the freshmen engineering class of 1938, which totaled 278 persons. The subjects were also found to be a representative group of the entire freshmen college population of that year.

Table 1 shows the product-moment coefficients and their probable errors as found between the seven primary abilities. Using this table as the correlation matrix, the centroid method of analysis was applied and four factors were extracted. The fourth factor was extracted only after rotation of three factors failed to give a clear interpretation of what those factors might be. With the addition of the fourth factor it was possible to produce a clearer picture for psychological interpretation.

Table 2 shows the residuals left after the fourth factor had been extracted. It will be seen from an examination of the columns that none of the configurations would have been thicker than .12. A projection of .12 after rotation would have been of little value in attempting to interpret reliably what the factor might be. It would appear then that a sufficient number of factors had been drawn.

The original factor loadings for the four factors extracted and their communalities are given in Table 3. After a process of some 20 rotations of the axes, it was felt that maximum simplification had been reached, since further rotations failed to change the picture obtained. Table 4 shows the factor loadings and their communalities after the last rotations had been made.

Interpretation of the Factors

The factor loadings were then examined in order to interpret

what the factors might be. The projections of Column I are all positive and rank in order as follows:

I	Induction	.6097
D	Reasoning	.5487
S	Space	.2668
N	Number	.1058
V	Verbal	.0156
P	Perception	.0087
M	Memory	.0036

Only two of the loadings in this column, those for the abilities of induction and reasoning, appear to be large enough to be considered as being significant for interpretation. Since these projections are for the reasoning and induction factors, it would seem that the factor operating here is reasoning. Whatever the true character of this factor may be, the evidence available at this time seems to indicate that there is an overlapping of the induction and reasoning factors. Thurstone (16) pointed out that his interpretation of these two factors was only tentative and there was need for further clarification of them. It will also be recalled that the highest correlations that have been reported between Thurstone's primary abilities, as far as the writer knows, have occurred in all instances but one between the reasoning and induction abilities, the exception being that reported by Ellison and Edgerton (2). Their highest correlation of .54 occurred between the primaries of space and perception.

Examination of Column II will reveal loadings of the following order:

M	Memory	.3843
N	Number	.1843
V	Verbal	.1607
S	Space	.0044
D	Reasoning	−.0139
I	Induction	−.0264
P	Perception	−.0836

The only loading that appears to be significant in this column is that of the memory ability. The loadings for the reasoning, induction, and perception abilities are negative in this column, but they are small enough to be considered as within the range of zero loadings. It may well be that the factor involved in this column is a specific of the memory tests.

Omitting Column III for the moment and turning to Column IV

it will be seen that the projections offer a rather clear picture. The loadings in order of size are as follows:

V	Verbal	.6036
D	Reasoning	.3566
P	Perceptual Speed	.3389
N	Number	.1509
I	Induction	.1287
S	Space	.0739
M	Memory	.0492

The highest loading above is that of verbal ability. Reasoning has the second highest loading, and examination of the three tests measuring this ability will disclose that one of the tests, the Arithmetic Test, is heavily weighted with verbal material. This may be the reason why the reasoning ability has a fairly high loading in this column. Perception ability has the third highest loading in this column and it will be found that of the two tests measuring this ability, one of them, the Verbal Enumeration Test, is also heavily weighted with verbal material. The remaining loadings are all positive but they are too small to be considered. It would seem that the factor operating here is a verbal factor.

Returning to Column III, which is the most interesting of the four disclosed, it will be seen that the factor loadings, given in order of size, are as follows:

S	Space	.6272
P	Perceptual Speed	.5709
N	Number	.5411
I	Induction	.3838
V	Verbal	.3101
M	Memory	.1524
D	Reasoning	.1004

The loadings in Column III are all positive, and with the exception of the reasoning and memory abilities, are all fairly high. The highest loading in this column is that of the space ability, yet to call this a space factor would be unwarranted, for it would be difficult to conceive of this space ability entering into the perception, number, induction, and verbal abilities. Examination of the rotations of this factor showed early in the process that the perception, space, number, induction, and verbal abilities were clinging close together and could not be separated. As was pointed out earlier, originally only three factors were rotated, and in an effort to break this clustering of these

abilities, a fourth factor was extracted. This helped to clarify the picture to some extent, since it was possible, after the fourth factor had been extracted, to rotate the axes so that the verbal factor was clearly disclosed on Axis IV; but despite the extraction of this fourth factor, the abilities still clustered together.

Throughout the whole process of rotation this clustering continued despite radical shifting of the axes. Originally these abilities were located on Axis II; these were then transferred by rotation to Axis IV, but this transferral failed to break down this clustering. Nor did consequent shifts of these loadings to Axes I, II, and III achieve isolation of these abilities. The only effect accomplished in these rotations was the change in order of size of the loadings of the abilities. After the first rotation the loadings of the abilities ranked in size in the following order: perception, number, verbal, space, memory, and induction. In the second and third rotations the order was perception, number, verbal, space, and induction. By the ninth rotation the order had changed to verbal, perception, number, space, and induction; and the final rotation placed them in the order of space, perception, number, induction, and verbal.

The entire development seems to point out that there is a close relationship among these abilities. As a result of this the writer is inclined to regard this factor as a *general factor* common to most of the abilities. This is of particular interest in light of the fact that these tests were designed to measure a number of pure abilities. Thurstone, (**16**) writing in 1939, stated: "So far in our work we have not found the general factor of Spearman, but our methods do not preclude it." In 1941 Thurstone and Thurstone (**18**), in a factorial study of the primary abilities of eighth-grade students, found what they have called a "second-order general factor." They call it so because, to quote them: "It makes its appearance, not as a separate factor, but as a factor inherent in the primaries and their correlations." Continuing, they state: "If further studies of the primary mental abilities of children should reveal this general factor, it will sustain Spearman's contention that there exists a general intellective factor . . . " (18, 26).

The finding of a general factor in this study seems to be in agreement with the Thurstones'. There is, however, this difference: the Thurstones' subjects were eighth-grade children and these writers appear to be of the opinion that this general factor might appear more strongly in younger children, as can be seen from this statement made by them: "It is now an interesting question to determine whether the correlations among primary abilities of still younger children will re-

veal, perhaps even more strongly, a second-order general factor" (18, 26).

It should be pointed out that the writer's disclosure of a *general factor* was found from the intercorrelations of scores made by college students whose average age was 19.58 years. It would appear from this evidence that this general factor is not restricted to younger-age groups since it has, in this case, been found for young adults.

TABLE 1

Intercorrelations of the Thurstone Primary Abilities and the Probable Errors for the Various Correlations

		N	V	S	M	I	D
P		+.32	+.40	+.41	+.05	+.29	+.13
		N	+.30	+.37	+.15	+.29	+.17
r	P.E.		V	+.24	+.17	+.21	+.29
.40–.49	.041			S	+.12	+.39	+.24
.30–.39	.046				M	+.10	+.03
.20–.29	.050					I	+.45
.00–.19	.052						D

TABLE 2

Table Showing the Residuals After the Fourth Factor Has Been Extracted

		+	1	+	2	−	3	+	4	−	5	+	6	+	7
+	1	+	.0674	−	.0220	−	.0289	+	.0289	+	.0264	+	.0227	−	.0674
+	2	−	.0220	+	.0220	−	.0130	−	.0090	+	.0132	+	.0033	+	.0074
−	3	−	.0289	−	.0130	+	.0480	+	.0063	+	.0347	−	.0058	−	.0480
+	4	+	.0289	−	.0090	+	.0063	+	.0289	−	.0141	−	.0202	−	.0121
−	5	+	.0264	+	.0132	+	.0347	−	.0141	+	.0398	−	.0398	−	.0202
+	6	+	.0227	+	.0033	−	.0058	−	.0202	−	.0398	+	.0398	+	.0163
+	7	−	.0674	+	.0074	−	.0480	−	.0121	−	.0202	+	.0163	+	.0674

TABLE 3

Table Showing the Original Factor Loadings and Their Communalities

Name of Factor	Original Factor Loadings				Communalities
	I	II	III	IV	h^2
Perception	.5587	.2738	−.1311	.2029	.445
Number	.5476	.1969	.1489	.0091	.360
Verbal	.5587	.1703	−.3442	.1716	.487
Space	.6060	.0818	.1958	.2405	.469
Memory	.2196	.1126	.1572	.2960	.173
Induction	.6060	−.3367	.1306	.1991	.537
Reasoning	.4892	−.4180	−.1594	.0204	.439

TABLE 4

Rotated Factor Loadings and Communalities

Name of Factor	Rotated Factor Loadings				Communalities
	I^{V}	II^{XII}	III^{X}	IV^{III}	h^2
Perception	.0087	−.0836	.5709	.3389	.447
Number	.1058	.1843	.5411	.1509	.360
Verbal	.0156	.1607	.3101	.6036	.486
Space	.2668	.0044	.6272	.0739	.469
Memory	.0036	.3843	.1524	.0492	.173
Induction	.6097	−.0264	.3838	.1287	.536
Reasoning	.5487	−.0139	.1004	.3566	.438

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A GENERAL THEORY OF LEARNING AND CONDITIONING: PART II

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The second of two parts of this article extends a mathematical theory of non-symbolic learning and conditioning to cases where reward and punishment are involved. The preceding results are generalized to the case where stimuli and responses are related psychophysically, thus constituting a theory of transfer, generalization, and discrimination.

4. *Instrumental conditioning and the effect of reward and punishment.*

In cases of what is often called "instrumental conditioning," in which the connection between stimulus and response is impressed at least partially by a reward following the evocation of the correct response, or is inhibited by a punishment so placed, a variety of observational considerations would appear to indicate that suitably placed affective stimuli influence the reaction-tendency E_j directly, in the manner of true conditioning, rather than merely the threshold R_j; thus, for example, inhibition of a conditioning consequent upon postliminary painful stimuli seems rather to have the nature of true counter-conditioning than inhibition, both in relatively slow temporal coarctation, and stability under influences which generally effect at least temporary loss of most inhibitions. Conversely, reinforcement by subsequent reward appears to have most of the properties of ordinary conditioning. This has been evidenced in several studies, especially those of Youtz (**55**, **56**),* and also Brogden, Lippman and Culler (**3**), Brogden and Culler (**2**), Bugelski (**5**), and Skinner (**46**), with a summary in (**18**). This being the case, when affective stimulation is involved in the experimental routine, we must modify the expression (10) to accommodate it.

Our procedure for deriving the effect of this factor will again be much like that of the previous sections. Considering the situation of

* The numbered references in the present paper are all to the bibliography in the previous section of the present discussion, *A General Theory of Learning and Conditioning, Part I*, which appeared in the March, 1943, issue of this journal.

section 1 again, let us add another interval $d\lambda$ after $d\eta$, containing a point λ, and denote the total affective stimulation at λ, reckoning rewards as positive and punishments negatively, by $V(\lambda)$. In human beings this may be determined psychophysically (Horst, 22), and in animals held constant and fitted indirectly from the data. We may designate by $\Delta_{\xi\eta\lambda}\, T_{ij}(\delta\,,\, t)$ the change in the ordinary increment $\Delta_{\xi\eta}\, T_{ij}(\delta\,,\, t)$ of T_{ij}, as given by (6), which results from the facts: (1) that at $d\xi$, there is type i stimulation of intensity $P_i(\xi)$; (2) at $d\eta$, there is conditioned type j response tendency derived from conditioning of magnitude $Q_j(\eta)$; while (3) at $d\lambda$ there is a quantity of affective stimulation $V(\lambda)$. Then we shall say

$$\Delta_{\xi\eta\lambda}\, T_{ij}(\delta\,, t) = V(\lambda)\; P_i(\xi)\; Q_j(\eta) \cdot e^{-a\delta-\beta^2(\eta-\xi-\delta)^2-\gamma(t-\lambda)-\varepsilon(\lambda-\eta)}.$$

The considerations leading up to the factor $P_i(\xi)\, e^{-a\delta-\beta^2(\eta-\xi-\delta)^2-\gamma(t-\lambda)}$ in this are the same as in (6), with the exception that here we have $\gamma(t-\lambda)$ instead of $\gamma(t-\eta)$ as before; this results from the fact that the increment (14) is not established until λ, and consequently cannot begin to decay until then. The factor $e^{-\varepsilon(\lambda-\eta)}$ arises from the necessity of some monotone decreasing factor, involving the distance $\lambda-\eta$, which is unity for zero separation and approaches zero asymptotically for increasing $\lambda-\eta$; it may perhaps be rendered plausible on the basis of some hypothesis involving the stimulus-trace of the conditioned part $Q_j(\eta)$, considered as a medium whereby the value of V at λ influences $T_{ij}(\delta\,,\, t)$ in a manner involving the value of Q_j at η. That the function of the distance $\lambda-\eta$ must have these properties follows from the experimentally observed diminution of the effect of affective stimulation as the interval between it and the reaction is augmented. The reason for using Q_j here in place of the perhaps more natural E_j, or possibly S_j, are partially observational: firstly, instrumental counter-conditioning by use of punishment can be set up in eductions of the reaction by the conditioned stimulus alone, in which case S_j vanishes throughout (Pavlov, 37 and Youtz, 55, 56);—this excludes the use of S_j for Q_j in (14)—moreover, we observe in general a resistance to instrumental counterconditioning which appears greater when the unconditioned reaction is very strong than it would seem to be under (14) with E_j for Q_j, for in this case the inhibition would be correspondingly more rapid in its rate of increase. This is of course not conclusive, but it appears to confer a presumption. A better founded decision must await more quantitative experimental evidence.

The expression (16) may be added to (5) and integrated over all

intervals $d\xi \to d\eta \to d\lambda$ which are concerned; and we obtain, upon a change in order of integration,

$$T_{ij}(\delta, t) = T^0_{ij}(\delta, t) + \int_0^t [S_j(\eta) - Q_j(\eta)]\, d\eta \int_0^\eta P_i(\xi) \times$$
$$e^{-\alpha\delta-\beta^2(\eta-\xi-\delta)^2-\gamma(t-\eta)}\, d\xi + \int_0^t Q_j(\eta)\, d\eta \int_\eta^t V(\lambda) \times \tag{17}$$
$$d\lambda \int_0^\eta P_i(\xi)\, e^{-\alpha\delta-\beta^2(\eta-\xi-\delta)^2-\gamma(t-\lambda)-\varepsilon(\lambda-\eta)}\, d\xi\,.$$

Equations (1) and (2) yield with this, if we remember (3), and again apply Dirichlet's rule,

$$Q_j(t) = \int_0^t \left\| \begin{matrix} \alpha, \beta, \gamma \\ \eta, t \end{matrix} \right\| [S_j(\eta) - Q_j(\eta)]\, d\eta + \int_0^t \left\| \begin{matrix} \alpha, \beta, \gamma \\ \varepsilon, \eta, t \end{matrix} \right\| Q_j(\eta)\, d\eta\,, \tag{18}$$

where we have set

$$\left\| \begin{matrix} \alpha, \beta, \gamma \\ \varepsilon, \eta, t \end{matrix} \right\| = \int_\eta^t d\theta \int_\eta^t V(\lambda)\, d\lambda \int_0^\eta F(\xi, \theta) \times$$
$$e^{-\alpha(t-\theta)-\beta^2(\eta-\xi-t+\theta)^2-\gamma(\theta-\lambda)-\varepsilon(\lambda-\eta)}\, d\xi\,. \tag{19}$$

Equation (18), like (10), is an integral equation to determine $Q_j(t)$, and, by (3) and (17), also E_j and T_{ij}. Since the same continuity conditions are satisfied as before, it is soluble by the standard methods, a process we may carry out in exact analogy to the solution of (10). Then we may put

$$K(t, \eta) = \left\| \begin{matrix} \alpha, \beta, \gamma \\ \varepsilon, \eta, t \end{matrix} \right\| - \left\| \begin{matrix} \alpha, \beta, \gamma \\ \eta, t \end{matrix} \right\|,$$

$$K^{(1)}(t, \eta) = K(t, \eta), \quad K^{(n+1)}(t, \eta) = \int_\eta^t K(\eta, \zeta) K^{(n)}(\zeta, t)\, d\zeta\,,$$

for all integers n, and

$$\Phi(t, \eta) = \sum_{n=1}^{\infty} K^{(n)}(t, \eta)\,.$$

Then similar manipulation to that of the previous case yields

$$Q_j(t) = \int_0^t \left\| \begin{matrix} \alpha, \beta, \gamma \\ \eta, t \end{matrix} \right\| S_j(\eta)\, d\eta + \int_0^t \int_0^\eta \Phi(t, \eta) \left\| \begin{matrix} \alpha, \beta, \gamma \\ \theta, \eta \end{matrix} \right\|$$
$$S_j(\theta)\, d\theta\, d\eta\,. \tag{20}$$

The computation of this solution in an actual case would be rather laborious, especially since the indicated quadratures in the expressions (9) and (19) for $\left\| \begin{matrix} \alpha, \beta, \gamma \\ \eta, t \end{matrix} \right\|$ and $\left\| \begin{matrix} \alpha, \beta, \gamma \\ \varepsilon, \eta, t \end{matrix} \right\|$ cannot be performed in closed form. We shall consider approximate solutions in a later study. Finally, we may remark that if the reaction considered is subject also to experimental extinction (55), the amount of this effect may be calculated from (15), but using the new value of Q_j.

5. The case of related stimuli and reaction-tendencies.

In the present section we shall sketch very briefly the extension of the results of previous sections to more general cases, where the values of functions involving one type of stimulus or reaction must be regarded as influencing the course of the learning process in quite different, although related, stimulus-response configurations. There is observational evidence to indicate that not only must association between stimuli—which results in generalization and transfer and the phenomena of discrimination—be taken into account, but also the relations between reactions, and, as a consequence of the latter, between reaction-thresholds. For this latter, e.g., Pavlov (37) has remarked that inhibition— which we treat as a rise in the reaction-threshold— is often generalized very widely, sufficiently so in fact to produce a sleep-like state in the organism. By proceeding in the same way as above, we shall be able to construct a theory capable of accounting for at least the comparative magnitudes of effects of the group mentioned.

We shall consider the various stimulus-and-reaction configurations involved in the situation as point-vectors in a psychophysical n-space, whose mutual distances may be taken for learning purposes as measuring the extent of their association. With human subjects we may locate the stimulus-and-reaction configurations in an appropriate vector space by the standard methods of modern psychophysics and factor analysis, scaling them so as to obtain their mutual distances in terms of "similarity" or "likeness," as estimated by the experimental population.

We shall suppose there to be some N stimulus-vectors $\bar{x}_i [i = 1, 2, \cdots, N]$, and M response-vectors, not necessarily in the same space as the $\bar{x}_i$, of such a kind that, if any stimulus-vector $\bar{x}$ has an unconditioned tendency to evoke a reaction-vector $\bar{y}$, this is the case solely because of its contiguity to some among the point-vectors $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N$, which have the property of evoking reaction-tendencies for some of the reaction-vectors $\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_M$ which are contiguous to $\bar{y}$. In

particular, if we define $T_o(\bar{x}, \bar{y}, \delta, t)$ as the amount of unconditioned response-tendency having the configuration $\bar{y}$ that a unit intensity of the stimulus-vector $\bar{x}$, presented at the time t, will evoke at $t + \delta$, we shall suppose that

$$T_o(\bar{x}, \bar{y}, \delta, t) = \sum_{i=1}^{N} \sum_{j=1}^{M} T_o(\bar{x}_i, \bar{z}_j, \delta, t)\, e^{-k^2[(\bar{x}_j - \bar{x})^2 + (\bar{z}_j - \bar{y})^2]}. \tag{21}$$

If $T(\bar{x}, \bar{y}, \delta, t)$ be the quantity of which $T_o(\bar{x}, \bar{y}, \delta, t)$ is the unconditioned part, we shall consider the above expression to reflect a more general fact: namely, that any increment $\Delta T(\bar{x}, \bar{y}, \delta, t)$ in the total value of the conditioning $T(\bar{x}, \bar{y}, \delta, t)$ for given vectors $\bar{x}, \bar{y}$ will bring to the value of $T(\bar{z}, \bar{w}, \delta, t)$ for like configurations $\bar{z}, \bar{w}$ an increment of strength

$$\Delta_{\bar{x}, \bar{y}} T(\bar{z}, \bar{w}, \delta, t) = \Delta T(\bar{x}, \bar{y}, \delta, t)\, e^{-k^2[(\bar{x} - \bar{z})^2 + (\bar{y} - \bar{w})^2]}. \tag{22}$$

Here, and correspondingly in (18), $(\bar{x} - \bar{z})^2$, $(\bar{y} - \bar{w})^2$ represent the squares of the magnitudes of the vectors $\bar{x} - \bar{z}$ and $\bar{y} - \bar{w}$, and are equal to the distances between the points whose radii vectores are respectively $\bar{x}$ and $\bar{z}$, or $\bar{y}$ and $\bar{w}$. The cases in the previous sections differ from our present ones in that all the stimuli and reactions occurring in the former are assumed to be infinitely far apart. Our choice of the normal distribution function to express the diminution of generalization with distance is consequent upon its simplicity as compared with other functions of the coordinates of the stimulus point-vectors which are unity with vanishing argument and approach zero asymptotically with increasing distance; but it may perhaps be also rendered plausible on the basis of a neurological hypothesis. There is evidence to indicate that generalization gradients are empirically at least rather like this: thus we may instance the results of Hovland (23), (24), and of Bass and Hull (1).

Let us denote by $\bar{x}(t)$ the stimulus configuration being presented at the time t; and by $P(t)$ the intensity of that stimulation. When $P(t)$ is zero, so that nothing is being presented, it will not matter what value is assigned to $\bar{x}(t)$. Further, let $E(\bar{y}, t)$ denote the intensity of the response-tendency at the time t to react with the response-configuration $\bar{y}$; the unconditioned part of this is

$$S(\bar{y}, t) = \int_0^t P(\theta) \sum_{i=1}^{N} \sum_{j=1}^{M} T_o(\bar{x}_i, \bar{z}_j, t - \theta, \theta)\, e^{-k^2[(\bar{x}_i - \bar{x}(\theta))]^2 + [\bar{z}_j - \bar{y}]^2}\, d\theta. \tag{23}$$

As in (2), we shall also have

$$E(\bar{y}, t) = \int_0^t P(\theta)\; T[x(\theta), y, t - \theta, \theta]\, d\theta\,. \tag{24}$$

We may now repeat an argument substantially like that leading up to (7), but taking into account (22) and a limiting process with respect to associated configurations. We derive

$$\begin{gathered} T(\bar{y}, \bar{z}, \delta, t) = T_0(\bar{y}, \bar{z}, \delta, t) + \int_V e^{-k^2(\bar{w}-\bar{y})^2}\, dV \\ \int_0^t [S(\bar{w}, \eta) - Q(\bar{w}, \eta)]\, d\eta \int_0^\eta P(\xi) \times \\ e^{-\alpha\delta-\beta^2(\eta-\xi-\delta)^2-\gamma(t-\eta)-k^2[\bar{x}(\xi)-\bar{y}]^2}\, d\xi\,, \end{gathered} \tag{25}$$

where the first integration on the right is a volume integral taken throughout the whole psychophysical space in which the response-configurations are located, $\bar{w}$ is the vector from the origin to the element of integration dV, and $Q_j = E_j - S_j$. Again, as before, we have from this

$$\begin{gathered} Q_j(\bar{y}, t) = \int_V e^{-k^2(\bar{w}-\bar{y})^2}\, dV \int_0^t [S(\bar{w}, \eta) - Q(\bar{w}, \eta)]\, d\eta \times \\ \int_\eta^t d\theta \int_0^\eta P(\xi)\, P(\theta) \\ e^{-\alpha(t-\theta)-\beta^2(\eta-\xi-t+\theta)^2-\gamma(t-\eta)-k^2[\bar{x}(\xi)-\bar{x}(\theta)]^2}\, d\xi\,. \end{gathered} \tag{26}$$

From this expression, precisely as before, $Q(\bar{y}, t)$ may be determined, although the practical derivation of the solution may be very difficult, since the volume integral in (26), if written out in terms of the coordinates of $\bar{w}$, would involve n successive integrations from $-\infty$ to ∞. The equation would be of mixed Fredholm-Volterra type.

We may proceed similarly to find the expression for $R(\bar{x}, t)$, the reaction-threshold for the response-configuration $\bar{x}$ at the time t, of which $R_0(\bar{x}, t)$ is the unconditioned part. Abbreviating

$$\begin{gathered} \left\| \begin{matrix} \alpha, \beta, \gamma, t \\ k, w, \eta, y \end{matrix} \right\| = \int_\eta^t d\theta \int_0^\eta P(\xi)\, P(\theta) \\ e^{-\alpha(t-\theta)-\beta^2(\eta-\xi-t+\theta)^2-\gamma(y, t-\eta)-k^2\{[\bar{x}(\xi)-\bar{x}(\theta)]^2+(\bar{w}-\bar{y})^2\}}\, d\xi\,, \end{gathered}$$

we shall have, analogous to (15),

$$R(\bar{y}, t) = R_0(\bar{y}, t) + \int_0^t d\eta \int_V \left\|\left| \begin{array}{c} \alpha, \beta', \gamma', t \\ k, \bar{w}, \eta, \bar{y} \end{array} \right|\right\| [S(\bar{w}, \eta) - Q(\bar{w}, \eta)]\, dV. \tag{27}$$

Finally, if reward and punishment be introduced, we shall have, giving $V(\lambda)$ the same meaning as before,

$$\begin{aligned} Q(\bar{y}, t) = {} & \int_0^t d\eta \int_V \left\|\left| \begin{array}{c} \alpha, \beta, \gamma, t \\ k, w, \eta, y \end{array} \right|\right\| S(\bar{w}, \eta)\, dV + \int_0^t d\eta \\ & \int_V Q(\bar{w}, \eta)\, dV \left\{ - \left\|\left| \begin{array}{c} \alpha, \beta, \gamma, t \\ k, w, \eta, y \end{array} \right|\right\| + \right. \\ & \int_\eta^t d\theta \int_\eta^t V(\lambda)\, d\lambda \int_0^\eta P(\xi)\, P(\theta) \\ & \left. e^{-\alpha(t-\theta) - \beta^2(\eta-\xi-t+\theta)^2 - \gamma(\theta-\lambda) - \varepsilon(\lambda-\eta) - k^2\{[\bar{x}(\xi) - \bar{x}(\theta)]^2 + (\bar{w}-\bar{y})^2\}}\, d\xi \right\}. \end{aligned} \tag{28}$$

On both of these equations, (27) and (28), V is the whole n-space, and $\bar{w}$ is again the radius vector to dV. For evidence, incidentally, that the situation of equation (28) actually occurs, i.e., that instrumental conditioning is in fact generalized, we may mention, among others Youtz (57), Münzinger and Dove (36), and Thorndike (51).

It is perhaps worthwhile to remark that certain previous theories of generalization, notably that of K. W. Spence, (48) and the elaboration thereupon by Hull (28), which have been strikingly confirmed in cases of "relational" versus "absolute" transfer as a basis for various phenomena in conditioned discrimination, emerge as special cases of the theoretical account given above, except that in accord with more recent data on the question, we agree with Razran (41) in preferring a negatively accelerated gradient to Spence's parabola. This difference is a minor one, however, and consequently the evidence of Gullikson (15), Gullikson and Wolfle (16), and Spence himself (47) in support of these hypotheses may also be used in favor of the present account.

6. Application to experimental situations.

The precise relation of the quantities we have calculated in the previous section to observation is something which must be stated in

rather greater detail than hitherto. We may, in general, distinguish three distinct types of situation with regard to the interpretation of our results: (1) The first of these may be represented by the conditioning experiment: here the subject may make any one of a number of mutually exclusive types of response, or, if his reaction-tendency for more of them is high enough, he may make no reaction of the group at all. (2) In the second type of situation, the subject is required to choose one among a certain group of mutually exclusive responses, so that failure to manifest any of them is not a possible alternative. For this type, we may choose as reference situation the case of the rat, which, when placed upon the platform of the learning apparatus, may jump either to the right or to the left, cannot do both, but must do one of them. (3) Thirdly, we may consider the case, properly falling under (1), where the response is not an all-or-none affair, but permits of quantitative variation in magnitude. Examples of this may be found in conditioning glandular or vasomotor reactions to various stimuli.

Suppose we define the *effective response-tendency* for responses of a given type j as

$$\mathcal{E}_j(t) = E_j(t) - R_j(t). \tag{29}$$

Now, in a situation of the first type considered above, let the effective reaction-tendencies at a given time t be $\mathcal{E}_j(t)\,[j = 1, \cdots, M]$. Consider the quantities $K_j(t) = \mathcal{E}_j(t) - \sum_{i \neq j} \mathcal{E}_i(t)$. At most, one of these quantities can be positive during a given interval. If none are, then we shall say that no reaction will occur during the interval. If, however, during an interval $t, t + \delta$, a given $K_j > 0$, then we shall say that a type j response will occur after a reaction-latency from t given by

$$L_j = m \log \frac{\log K_j - q}{\log K_j - (q + r)}, \tag{30}$$

for suitable constants m, q, r. The equation (27) is one derived by Landahl for reaction-times from neurological considerations (see Rashevsky, 40); it appears to fit well to observations. This assertion gives us an account of the observed interference of competing reaction-tendencies at the same time which seems to be in accord with the principal facts in the literature about the subject: see, e.g., Hilgard and Marquis (18), Hull (27), and investigations such as that of Kellogg and Wolf (33).

It is worth remarking that the statement above, together with other similar ones in the present section, does not constitute a new

assumption over and above those we have made before; it has, in fact, the character of a definition. It fixes, although not completely, the meaning of the functions E_j and R_j which we introduced above without defining them formally, trusting to the intuition of the reader to supply them with enough meaning to enable him to follow the arguments adduced from experiment. For a theory of quasi-definitions of this sort (they are called *reduction-sentences*, in that they enable us to reduce certain assertions involving the definienda—although not all, as would be the case with explicit definitions—to sentences which are directly confirmable by observation), we refer the reader to various works of R. Carnap, in particular (8) and (9). Quasi-definitions of this kind are very common in science: see, e.g., the physical definition of "electric intensity."

In the second type of situation mentioned, we clearly cannot use the same method of predicting responses: if all the K_j are negative, we infer from the above that nothing will be evoked; but by hypothesis, even in this case some reaction must occur. In this case, we shall regard the subliminal reaction-tendencies $\mathcal{E}_j$ as representing probabilities of the corresponding types of response; and we shall say that, if at any time t the subject is required to make a response of one of several mutually exclusive types $1, 2, \cdots, M$, then the probability of a response of a given type j is

$$P_j = \frac{\mathcal{E}_j}{\sum_{i=1}^{M} \mathcal{E}_i}. \tag{31}$$

The meaning of the $\mathcal{E}_j$ in this case seems to be rather different from that in type (1) situations; but there does not appear to be a simple way of remedying this, since any attempt to represent the $\mathcal{E}_j$ as probabilities in type (1) situations encounters the difficulty of providing a plausible value for the probability of no response. Moreover, type (2) situations can hardly avoid being much more under voluntary control; and by this route a number of disregarded factors may become important, to hold which constant a statistical procedure of gathering data and interpreting our formulae may be made necessary.

Situations of the third type considered above present a somewhat different sort of problem. We have so far tended to regard $\mathcal{E}_j$ as possessing the dimensions of an average strength of neural stimulation, and the relation of such strength of excitation to the intensity of resulting responses capable of varying magnitudes is not very well known. Extremely tentatively, however, we may perhaps assume the relationship to be a linear one: $I_j = k\,\mathcal{E}_j$ for fairly small positive values of $\mathcal{E}_j$,

or else $I_j = k(1 - e^{-\mu \mathcal{E}_j})$ for positive $\mathcal{E}_j$ and suitable k, μ. The whole matter requires further experimental study.

By way of conclusion we may make several remarks. First, our introductory purpose must be considered fulfilled, since we have devised a theory of sufficient generality to provide an experimental solution for all those cases of learning and conditioning which we set out to consider. Second, our theory agrees qualitatively with a sufficient number of experimentally observed effects to make a *prima facie* case for its being, if not true, at least not far from the truth at many points, so that it should merit careful experimental study. In fact, of the well-known observational effects, there seem to be only two which we can say at present are clearly in disagreement with the theory: namely, that the facility of experimental extinction is apparently much reduced when courses of extinction have been interpolated in the conditioning process; and that the process of "second-order conditioning," in which a third stimulus is conditioned to a reaction by use, in place of an unconditioned stimulus, of one previously conditioned to the reaction, although rather difficult to bring about, nevertheless occurs, whereas under our theory this process would only produce conditioned inhibition—unless affective elements are involved, as in the experiments of Brogden, Lippman, and Culler (3), and Brogden and Culler (2), where we should derive the proper prediction. We shall attempt to account for such cases, and in addition examine the possibilities for grounding the theory neurologically, in a later study, where we shall also make quantitative comparison of the theory with whatever precise data are available, and attempt to approximate some of the more complex and involved functions in the theory in a simpler manner. Meanwhile, the reader is invited to compare the theory with any suitable experimental results.